G. F. Khilmi

Qualitative Methods in the Many Body Problem

$$\frac{d^2\xi_i}{dt^2} = -m_0 \frac{\xi_i}{r_{0i}^3} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \frac{m_j(\xi_i - \xi_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \xi_j}{r_{j0}^3}$$

$$\frac{d^2\eta_i}{dt^2} = -m_0 \frac{\eta_i}{r_{0i}^3} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \frac{m_j(\eta_i - \eta_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \eta_j}{r_{j0}^3}$$

$$\frac{d^2\zeta_i}{dt^2} = -m_0 \frac{\zeta_i}{r_{0i}^3} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \frac{m_j(\zeta_i - \zeta_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \zeta_j}{r_{j0}^3}$$

$$i = 1, 2, \ldots, n-1$$

Qualitative Methods in the Many Body Problem

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Introduction

1. The n-body problem is the name usually given to the problem of the motion of a system of many particles attracting each other according to Newton's law of gravitation. This is the classical problem of mathematical natural science, the significance of which goes far beyond the limits of its astronomical applications.

The n-body problem has been the main topic of celestial mechanics from the time of its inception as a science. Now that many of the problems of celestial mechanics have become part of geophysics, the central position of the n-body problem has been further strengthened.

The fundamental dynamical problem for a system of n gravitating bodies is the investigation and predetermination of the changes in position and velocity that the particles undergo as the time varies. However, this is a very complex non-linear problem whose solution has not been possible under the present-day status of mathematical analysis.

It is natural that first to be developed were the practical aspects of the n-body problem, arising in connection with the study of the motion of the planets, comets, and satellites. These problems were solved by the method of successive approximations. However, the mathematical difficulties entailed by these problems were so great, that the convergence of the approximations was not proved nor an estimate of the error obtained. Although the computational methods were not rigorously justified, their applicability, nevertheless, has been verified by practical usage which has shown that there exists an amazing agreement between theory and observation in most of the cases

considered. Despite their mathematical imperfection, the practical methods of celestial mechanics yield an accurate picture of the motion of the celestial bodies over a limited interval of time.

2. Today, the hope for achieving some success in the study of the n-body problem is centered mainly on the use of *qualitative* methods of investigation.

A qualitative investigation of the n-body problem usually consists in studying an incomplete set of parameters which, though not defining the state of the system uniquely, nevertheless, characterizes it in certain respects. Clearly, the problem of the predetermination of the values of an incomplete set of parameters cannot be well-posed, and we must confine ourselves to ascertaining the general laws by which these parameters change. This restricted formulation of the problem, characteristic of qualitative investigations, has the advantage that a knowledge of all the integrals of the motion is not needed for its solution, and significant results can be obtained by using only the known integrals, or by analyzing directly the equations of motion and their various analytical consequences.

However, in choosing an incomplete set of parameters, it is important that we choose a representative set capable of reflecting the essential characteristics of the motion, and which leads to meaningful ideas about the dynamical system. To the set of such parameters belong, first of all, the mutual distances between the bodies of the system. It is therefore not accidental, that the search for the most general properties of the distances between the bodies as the time tends to infinity (positively or negatively) has occupied a prominent place in the qualitative investigations of celestial mechanics. This problem is often called the "problem of final motions".

The first general and very simple results on the final motion of n gravitating bodies were obtained by Jacobi in 1842. However, after Jacobi's work, attempts were made to study only the special case of three bodies. A systematic investigation of the final motion of three gravitating bodies was begun about thirty years ago by J. Chazy. For this case, he attempted to classify the types of motions possible and to investigate their connection.

The results of Chazy proved to be very interesting, although one of his most important ones, the impossibility of capture, is incorrect. After O.Yu. Schmidt showed, with the help of the techniques of numerical integration, that capture is possible, the need arose to carry on a more detailed study of the final motion in the three-body problem. It was necessary to obtain effective qualitative theorems that give sufficient conditions for the occurrence of specific types of motions in the form of restrictions on the initial data. A general approach to the solution of this problem was presented by the author in the years 1948-1951, and he obtained the very first criteria for the occurrence of hyperbolic and hyperbolic-elliptic type motions in the three-body problem.

Research in this direction was continued by G.A. Merman, K.A. Sitnikov, V.F. Proskurin, N.G. Kochina, G. Ye. Khrapovitskaya, and O.A. Sizova. Thus, by today, a considerable amount of scientific work on this question has been accumulated (see the bibliography at the end of the book).

We stated above that after Jacobi's investigations, no attempts were made to obtain new results on the character of the final motion in the general case of an arbitrary number of bodies. The mathematical difficulties of this problem have apparently caused researchers to stay away from it. Nevertheless, we believe that progress in the

general case is possible, although it seems it may not be rapid.

Below, we are going to consider some qualitative methods of analyzing the n-body problem, as well as some of the results obtained by means of these methods. Of course, the totality of these results does not, by far, give a complete qualitative picture of the motion of n gravitating bodies, and its clarification will require still further work.

3. Let us give a brief outline of the content of the book.

In the first chapter, we give the equations and general integrals of the n-body problem, and we study the simplest theorems on the final motion due to Jacobi.

In the second chapter, we consider means of applying the method of dimensional analysis to the n-body problem. As far as we know, dimensional analysis has not been used before in the investigation of the final motion; though it does not yield definitive results in this area, it is very useful in carrying out a preliminary analysis of the problem.

In the third chapter, we present our "method of continuous induction," and we consider some applications of it in which the final motion in the n-body problem is analyzed. This method allows us to obtain effective qualitative results, namely, it allows us to formulate sufficient conditions for the occurrence of certain types of final motions in the form of conditions on the initial state of the dynamical system.

The fourth chapter is devoted to the method of invariant measure. The application of this method to the many-body problem has necessitated working out a number of theorems on the measure theory of dynamical systems. These are presented at the beginning of the chapter. At the end of the chapter, we prove some very

general theorems on the motion of a system of gravitating bodies using the method of invariant measure.

In the fifth chapter, an attempt is made to analyze some cases of the evolution of a system of n gravitating bodies on the basis of celestial mechanics. Here, we shall be concerned with the processes which are of cosmogonical interest, and which are accompanied by the conversion of mechanical energy into non-mechanical forms.

Contents

CHAPTER 1

Equations and General Integrals of the n-Body Problem. Simplest Theorems Concerning the Final Motion.

1. We shall consider a system of n particles P_1, P_2, $\ldots$, P_n with masses m_1, m_2, $\ldots$, m_n attracting each other according to Newton's law of gravitation. We shall study the relative motion of these particles. At first, we consider the motion of P_1, P_2, $\ldots$, P_n with respect to the center of mass of the system.

Let us take a set of rectangular coordinates with origin fixed at the center of mass of the system and with coordinate axes having fixed directions. Let x_i, y_i, z_i be the coordinates of the point P_i, r_i the distance of P_i from the origin, and r_{ij} the distance between P_i and P_j. We shall use a system of units for which the universal gravitational constant is 1.

Let us set

$$U = \sum_{\substack{ij \\ i \neq j}} \frac{m_i m_j}{r_{ij}}, \tag{1.1}$$

where the summation is taken over all pairs of distinct bodies; then the equations of motion are given by the following $3n$ differential equations of second order:

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}. \qquad (1.2)$$

$$i = 1, 2, \ldots, n$$

(In this chapter, we shall only consider the equations of motion and their integrals.) These equations may also be written as $6n$ equations of first order

$$\frac{dx_i}{dt} = x'_i, \quad \frac{dy_i}{dt} = y'_i, \quad \frac{dz_i}{dt} = z'_i,$$

$$m_i \frac{dx'_i}{dt} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{dy'_i}{dt} = \frac{\partial U}{\partial y_i}. \quad m_i \frac{dz'_i}{dt} = \frac{\partial U}{\partial z_i}. \qquad (1.3)$$

$$i = 1, 2, \ldots, n$$

A first integral of these equations is the familiar law of conservation of energy

$$\frac{1}{2} \sum_{i=1}^{n} m_i (x_i'^2 + y_i'^2 + z_i'^2) = U + H. \qquad (1.4)$$

where H is the constant of integration.

Besides this, there are the six linear momentum integrals. In our choice of coordinates they are given by

$$\sum_{i=1}^{n} m_i x'_i = 0, \quad \sum_{i=1}^{n} m_i y'_i = 0, \quad \sum_{i=1}^{n} m_i z'_i = 0,$$

$$\sum_{i=1}^{n} m_i x_i = 0, \quad \sum_{i=1}^{n} m_i y_i = 0, \quad \sum_{i=1}^{n} m_i z_i = 0. \qquad (1.5)$$

There are still the three area (moment of momentum) integrals which, in terms of the coordinates under consideration, are

$$\begin{aligned}
&\sum_{i=1}^{n} m_i (x_i y_i' - y_i x_i') = c_1, \\
&\sum_{i=1}^{n} m_i (y_i z_i' - z_i y_i') = c_2, \qquad (1.6) \\
&\sum_{i=1}^{n} m_i (z_i x_i' - x_i z_i') = c_3,
\end{aligned}$$

Here c_1, c_2, and c_3 are constants of integration.

These ten integrals exhaust all of the known essentially independent integrals of the given dynamical system.

2. Another theoretically useful way of describing the relative motion of n bodies can be given by introducing Jacobi coordinates. We first change the notation of the preceding section into one that is better suited to the application of Jacobi coordinates.

Let the particles in question now be denoted by P_0, $P_1, \ldots, P_{n-1}$ and their respective masses by $m_0, m_1, \ldots,$ m_{n-1}.

Let ξ_1, η_1, ζ_1, be the coordinates of P_1 with respect to axes with origin at the point P_0, ξ_2, η_2, ζ_2 the coordinates of P_2 with respect to axes with origin at the center of mass of the particles P_0 and P_1 etc., and finally, let ξ_{n-1}, η_{n-1}, ζ_{n-1} be the coordinates of the point P_{n-1} with respect to axes with origin at the center of mass of the particles P_0, $P_1, \ldots, P_{n-2}$. The coordinate axes in all of the systems are assumed to be parallel. Let ρ_i denote the distance of P_i from the origin of the coordinate system in which the motion of this particle is being described. As before, we let r_{ij} denote the distance between P_i and P_j, and we again use a system of units in which the universal gravitational constant is 1.

The equations of motion in Jacobi coordinates are then given by

$$\mu_i \frac{d^2\xi_i}{dt^2} = \frac{\partial U}{\partial \xi_i}, \quad \mu_i \frac{d^2\eta_i}{dt^2} = \frac{\partial U}{\partial \eta_i}, \quad \mu_i \frac{d^2\zeta_i}{dt^2} = \frac{\partial U}{\partial \zeta_i}, \qquad (1.7)$$

$$i = 1, 2, \ldots, n-1$$

where $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the reduced masses defined by

$$\mu_1 = \frac{m_0 m_1}{m_0 + m_1},$$

$$\mu_2 = \frac{(m_0 + m_1) m_2}{m_0 + m_1 + m_2}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mu_{n-1} = \frac{(m_0 + m_1 + \ldots + m_{n-2}) m_{n-1}}{m_0 + m_1 + \ldots + m_{n-1}}.$$

Equations (1.7) may be expressed as $6(n-1)$ equations of the first order:

$$\frac{d\xi_i}{dt} = \xi_i', \quad \frac{d\eta_i}{dt} = \eta_i', \quad \frac{d\zeta_i}{dt} = \zeta_i',$$

$$\mu_i \frac{d\xi_i'}{dt} = \frac{\partial U}{\partial \xi_i}, \quad \mu_i \frac{d\eta_i'}{dt} = \frac{\partial U}{\partial \eta_i}, \quad \mu_i \frac{d\zeta_i'}{dt} = \frac{\partial U}{\partial \zeta_i}. \qquad (1.8)$$

$$i = 1, 2, \ldots, n-1$$

In Jacobi coordinates, the energy integral becomes

$$\frac{1}{2} \sum_{i=1}^{n-1} \mu_i (\xi_i'^2 + \eta_i'^2 + \zeta_i'^2) = U + H, \qquad (1.9)$$

where H is the constant of integration.

The momentum integrals are satisfied identically in Jacobi coordinates, and the moment of momentum integrals take on the form

$$
\begin{aligned}
&\sum_{i=1}^{n-1} \mu_i (\xi_i \eta_i' - \eta_i \xi_i') = c_1, \\
&\sum_{i=1}^{n-1} \mu_i (\eta_i \zeta_i' - \zeta_i \eta_i') = c_2, \qquad (1.10) \\
&\sum_{i=1}^{n-1} \mu_i (\zeta_i \xi_i' - \xi_i \zeta_i') = c_3,
\end{aligned}
$$

where c_1, c_2, and c_3 are constants of integration.

3. Finally, we give one more way of describing the relative motion of n gravitating bodies $P_0, P_1, \ldots, P_{n-1}$ using a coordinate system with origin located at one of the points, for instance, the point P_0.

The equations of motion in this case are given by

$$
\begin{aligned}
\frac{d^2\xi_i}{dt^2} &= -m_0 \frac{\xi_i}{r_{0i}^3} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \frac{m_j(\xi_i - \xi_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \xi_j}{r_{j0}^3}, \\
\frac{d^2\eta_i}{dt^2} &= -m_0 \frac{\eta_i}{r_{0i}^3} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \frac{m_j(\eta_i - \eta_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \eta_j}{r_{j0}^3}, \qquad (1.11) \\
\frac{d^2\zeta_i}{dt^2} &= -m_0 \frac{\zeta_i}{r_{0i}^3} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \frac{m_j(\zeta_i - \zeta_j)}{r_{ij}^3} - \sum_{j=1}^{n-1} \frac{m_j \zeta_j}{r_{j0}^3}, \\
& i = 1, 2, \ldots, n-1
\end{aligned}
$$

where ξ_i, η_i, ζ_i are the coordinates of P_i.

We shall also find it convenient to write these equations in vector form. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors directed along

the ξ, η, ζ axes, respectively, and $\mathbf{r}_{ij}$ the vector joining P_i to P_j. Then multiplying equations (1.11) respectively by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and adding, we obtain

$$\frac{d^2\mathbf{r}_{i0}}{dt^2} = -m_0\frac{\mathbf{r}_{i0}}{r_{i0}^3} + \sum_{\substack{j=1\\ j\neq i}}^{n-1} m_j\frac{\mathbf{r}_{ij}}{r_{ij}^3} - \sum_{j=1}^{n-1} m_j\frac{\mathbf{r}_{j0}}{r_{j0}^3}\,. \qquad i=1, 2, \ldots, n-1$$

We have just considered the motion of the system of particles relative to P_0. However, we could consider the motion of all the particles relative to any particle P_k. We may therefore write the following equations:

$$\frac{d^2\mathbf{r}_{ik}}{dt^2} = -m_k\frac{\mathbf{r}_{ik}}{r_{ik}^3} + \sum_{\substack{j=0\\ j\neq i, k}}^{n-1} m_j\frac{\mathbf{r}_{ij}}{r_{ij}^3} - \sum_{\substack{j=1\\ j\neq k}}^{n-1} m_j\frac{\mathbf{r}_{jk}}{r_{jk}^3}\,. \qquad \begin{array}{c} i=0, 1, \ldots, n-1\\ k=0, 1, \ldots, n-1\\ i\neq k\end{array} \tag{1.12}$$

4. Let us now consider the moment of inertia of n gravitating particles:

$$J^2 = \sum_{i=1}^{n} m_i r_i^2. \tag{1.13}$$

Differentiating this twice with respect to the time t and taking into account that

$$r_i^2 = x_i^2 + y_i^2 + z_i^2,$$

we obtain

$$\frac{d^2J^2}{dt^2} = 2\sum_{i=1}^{n} m_i(x_i x_i'' + y_i y_i'' + z_i z_i'' + x_i'^2 + y_i'^2 + z_i'^2).$$

By making use of (1.2) and (1.4), we can write this equation as follows:

$$\frac{d^2J^2}{dt^2} = 2\sum_{i=1}^{n}\left(x_i\frac{\partial U}{\partial x_i} + y_i\frac{\partial U}{\partial y_i} + z_i\frac{\partial U}{\partial z_i}\right) + 4(U+H).$$

Now, the potential function U is a homogeneous function of the coordinates of degree -1; therefore

$$\sum_{i=1}^{n}\left(x_i\frac{\partial U}{\partial x_i} + y_i\frac{\partial U}{\partial y_i} + z_i\frac{\partial U}{\partial z_i}\right) = -U$$

and hence,

$$\frac{d^2J^2}{dt^2} = 2(U+2H). \tag{1.14}$$

This equation is called the Lagrange-Jacobi equation and plays an important role in celestial mechanics. It was obtained by Lagrange in the case of three bodies and was generalized to n bodies by Jacobi.

5. We next derive an auxiliary equation which allows us to express the moment of inertia in terms of the distances between the bodies.

Consider the distances of the bodies $P_1, P_2, \ldots, P_n$ from the center of mass of the system and the distances between the bodies. The square of the distance of P_i from the center of mass is

$$r_i^2 = x_i^2 + y_i^2 + z_i^2, \tag{1.15}$$

and the square of the distance between P_i and P_j is

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2. \tag{1.16}$$

We previously used $\sum_{ij}$ to stand for the summation over all distinct pairs ij ($i \neq j$ and ji not considered to be distinct from ij); we now let $\sum^*_{ij}$ denote the summation over all possible pairs ij (any i is taken in combination with any j).

We then have the following obvious equations:

$$\sum_{ij} m_i m_j (x_i - x_j)^2 = \sum_{ij}^{*} m_i m_j (x_i - x_j)^2 =$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j (x_i - x_j)^2 =$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j x_i^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j x_j^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j x_i x_j =$$

$$= \frac{M}{2} \sum_{i=1}^{n} m_i x_i^2 + \frac{M}{2} \sum_{j=1}^{n} m_j x_j^2 - \sum_{i=1}^{n} m_i x_i \sum_{j=1}^{n} m_j x_j,$$

where

$$M = \sum_{i=1}^{n} m_i.$$

However, since

$$\sum_{i=1}^{n} m_i x_i^2 = \sum_{j=1}^{n} m_j x_j^2,$$

and

$$\sum_{i=1}^{n} m_i x_i = \sum_{j=1}^{n} m_j x_j = 0,$$

by the linear momentum integrals, we find from the above equations that

$$M \sum_{i=1}^{n} m_i x_i^2 = \sum_{ij} m_i m_j (x_i - x_j)^2. \qquad (1.17a)$$

In an analogous way, we obtain the formulas

$$M \sum_{i=1}^{n} m_i y_i^2 = \sum_{ij} m_i m_j (y_i - y_j)^2, \qquad (1.17b)$$

$$M \sum_{i=1}^{n} m_i z_i^2 = \sum_{ij} m_i m_j (z_i - z_j)^2. \qquad (1.17c)$$

Adding (1.17a), (1.17b) and (1.17c) and taking into account (1.15) and (1.16), we find

$$M \sum_{i=1}^{n} m_i r_i^2 = \sum_{ij} m_i m_j r_{ij}^2. \qquad (1.18)$$

Let us introduce the notation

$$I^2 = \sum_{ij} m_i m_j r_{ij}^2.$$

Then by (1.13) and (1.18), we have

$$J^2 = \frac{1}{M} I^2.$$

Thus, the Lagrange-Jacobi equation is expressible in the form

$$\frac{1}{M}\frac{d^2I^2}{dt^2}=2(U+2H). \tag{1.19}$$

6. The energy integral and Lagrange-Jacobi equation permit us to establish certain properties of the distances r_{ij} as time becomes negatively or positively infinite, without the use of additional analytical tools, for the general case of a system of n gravitating bodies. It is true that in this way we obtain only the simplest theorems on the final motion which already were given by Jacobi. However, the great importance of these theorems is that they reveal the proper way to formulate the problem of final motions, as well as indicate what general course a more detailed investigation should take.

We first consider the properties of the distances r_{ij} in a system of n gravitating bodies with a negative total energy.

THEOREM 1.1. *If the constant H in the law of conservation of energy is negative, and if as $t\to\infty$ $(t\to-\infty)$ no collisions occur between the bodies, then for all $t>0$ $(t<0)$ the function*

$$r_*(t)=\min_{ij}\{r_{ij}\}$$

possesses the following properties:

(1) *For all $t>0(t<0)$*

$$r_*(t)\leqslant\frac{M^2}{2|H|}.$$

(2) *For any $T>0$ no matter how large and any $\epsilon>0$ no matter how small, there exists a time t' with $|t'|\geqslant T$ such that*

$$r_*(t') < \frac{M^2}{4|H| - 2\varepsilon},$$

where

$$M = \sum_{i=1}^{n} m_i.$$

Let us prove the first statement. Since the kinetic energy of the system cannot be negative, it follows from the conservation law that

$$U + H \geqslant 0$$

or

$$\sum_{ij} \frac{m_i m_j}{r_{ij}} \geqslant |H|.$$

We can strengthen this inequality and write

$$\frac{1}{r_*(t)} \sum_{ij} m_i m_j \geqslant |H|. \tag{1.20}$$

However, it is not difficult to see that

$$\sum_{ij} m_i m_j \leqslant \frac{M^2}{2}.$$

Thus without weakening inequality (1.20), we may write

$$\frac{M^2}{2r_*(t)} \geqslant |H|,$$

and hence

$$r_*(t) \leqslant \frac{M^2}{2|H|}. \tag{1.21}$$

We now prove the second statement. We show that for any positive T, no matter how large, we can find a $t' > T$ for which

$$U + 2H > -\epsilon.$$

Assume the contrary. Then for some value of T we have

$$U + 2H \leqslant -\epsilon \tag{1.22}$$

for all values of $t > T$.

Integrating the Lagrange-Jacobi equation (1.19) twice with respect to time from T up to $t > T$, we obtain

$$I^2(t) = I^2(T) + I^{2\prime}(T)(t-T) + 2M \int_T^t \int_T^t (U + 2H)\, dt\, dt.$$

and hence by (1.22)

$$I^2(t) \leqslant I^2(T) + I^{2\prime}(T)(t-T) - M\varepsilon (t-T)^2.$$

We see from this that $I^2(t) \to -\infty$ as $t \to +\infty$. Therefore there must exist a time $\tau > 0$ such that $I^2(\tau) = 0$, i.e., a collision between the bodies occurs. This contradicts the condition of the theorem.

Thus, there exists a $t' > T$ for which

$$U(t') + 2H > -\epsilon,$$

$$U(t') > 2|H| - \epsilon.$$

Strengthening this inequality, we have

$$\frac{M^2}{2r_*(t')} > |H| - \epsilon,$$

or

$$r_*(t') < \frac{M^2}{4|H|-2\epsilon},$$

and so for $t \geqslant 0$ the theorem is proved. For $t \leqslant 0$, the theorem is proved in a similar way.

The fact that when $H<0$, some of the distances can increase indefinitely with the time, does not contradict the theorem just proved. However, it follows from the theorem that this cannot happen to all the distances. Thus whenever $H<0$, it is impossible for a system of gravitating bodies to be completely scattered either as $t \to \infty$ or as $t \to -\infty$.

We now consider a system of gravitating bodies for which the total energy is positive, i.e., $H>0$.

THEOREM 1.2. (Jacobi's Theorem). *If the constant H in the energy integral is positive, and if as $t \to \infty$ ($t \to -\infty$) no collisions occur between the bodies, then at least one of the distances r_{ij} increases indefinitely.*

Proof: We start with the Lagrange-Jacobi equation in the form (1.19) and integrate it twice with respect to time from $t=0$ to $t>0$. We thus obtain

$$I^2(t) = I_0^2 + I_0^{2\prime} t + 2M \int_0^t \int_0^t (U + 2H)\, dt\, dt,$$

where I_0^2 and $I_0^{2\prime}$ are the respective values of I^2 and $I^{2\prime}$ at $t=0$. Now, since $U>0$, it follows that

$$I^2(t) > I_0^2 + I_0^{2\prime} t + 4M \int_0^t \int_0^t H\, dt\, dt,$$

or

$$I^2(t) > I_0^2 + I_0^{2\prime} t + 2MHt^2.$$

When $H>0$, we find from this inequality that

$$I^2(t) \to \infty \text{ as } t \to +\infty \quad (t \to -\infty),$$

This is only possible if at least one of the distances r_{ij} increases indefinitely as $t \to +\infty$ $(t \to -\infty)$. The theorem is proved.

This theorem is not incompatible with the fact that, when $H>0$, instances are possible in which all of the distances between the bodies increase indefinitely as $t \to \infty$, or as $t \to -\infty$, or else in both cases. However, the theorem is also not incompatible with the fact that, when $H>0$, there may be subsystems in which the distances between the bodies remain bounded at $t \to \infty$, or as $t \to -\infty$, or as t approaches both $\pm\infty$. Finally, the fact that there may be complete scattering of the system as the time increases in one direction coupled with incomplete scattering as the time increases in the opposite direction is not incompatible with the theorem.

In conclusion, by comparing Theorems 1.1 and 1.2, we can state that, when $H<0$, the energy in the system is insufficient for it to be completely scattered, and when $H>0$, the energy is so high, that at least a partial scattering of the system is inevitable.

CHAPTER 2

Method of Dimensional Analysis

1. The method of dimensional analysis, which has lately been applied rather widely in diversified problems of mechanics and physics, can also be used in the qualitative analysis of the motion of systems of gravitating bodies. However, it must be borne in mind that dimensional analysis is incapable of yielding complete enough results for the n-body problem, though in certain instances it enables important conclusions to be obtained.

Our purpose is to consider means of applying the methods of dimensional analysis to find criteria for the occurrence of various types of motions in a system of n gravitating bodies. In those instances where dimensional analysis enables us to obtain such a criterion, we find that it is expressible exactly to within an unknown dimensionless factor. Thus, the methods of dimensional analysis do not lead to a complete criterion, but only give an indication of its analytical form. Nevertheless, such restricted results are of importance since they yield preliminary knowledge and give us a correct orientation at the initial stages of our investigation.

2. That dimensional analysis is capable of indicating the analytical form of the dependence of two dimensional quantities springs from the fact that objective physical relationships cannot depend on the choice of the system of units. However, such relationships between quantities are not arbitrary and are determined by the dimensions of these quantities.

Dimensional analysis is usually applied to some relation expressible in the form of an equation or identity.

What is somewhat new in the application of the method of dimensional analysis to the qualitative study of the n-body problem is the necessity for considering inequalities. For our further purposes, the following reasoning will suffice.

(1) Let $\mathfrak{M}$ be a set of systems of basic independent units

$$q_1, q_2, \ldots, q_n$$

such that a transformation to any other system of units

$$q'_1, q'_2, \ldots, q'_n,$$

also belonging to $\mathfrak{M}$, is effected by means of the formulas

$$q'_i = \alpha_i q_i, \quad i = 1, 2, \ldots, n \tag{2.1}$$

where α_i are arbitrary positive numbers.

(2) There exist arbitrary positive quantities

$$Q_0, Q_1, Q_2, \ldots, Q_n,$$

which are *homogeneous* in the sense that the values of these quantities:

$$Q'_0, Q'_1, Q'_2, \ldots, Q'_n$$

after transforming the basic units according to (2.1), are related to their previous values by the equations

$$Q'_j = \alpha_1^{a_{j1}} \alpha_2^{a_{j2}} \ldots \alpha_n^{a_{jn}} Q_j. \tag{2.2}$$

The exponents a_{jk} are the powers of the dimensions; if they are all zero, then the Q_j are dimensionless. It is clear that the product of homogeneous quantities is again homogeneous.

(3) The quantity Q_0 is related to $Q_1, Q_2, \ldots, Q_n$ by the inequality

$$Q_0 \leqslant f(Q_1, Q_2, \ldots, Q_n). \tag{2.3}$$

(4) The relation $Q_0 \leqslant f(Q_1, Q_2, \ldots, Q_n)$ *does not depend on the units of measurement* in the sense it is satisfied after every transformation of the basic units by means of (2.1).

(5) All the basic units enter into the expressions for the quantities $Q_0, Q_1, \ldots, Q_n$.

THEOREM 2.1. *If conditions* (1) *through* (5) *are satisfied, then we have*

$$Q_0 \leqslant C Q_1^{x_1} Q_2^{x_2} \ldots Q_n^{x_n}, \tag{2.4}$$

where the dimensionless constant $C = f(1, 1, \ldots, 1)$, *and the exponents* x_j *are uniquely determined by condition* (4).

Proof: We take the quantities $Q_1, Q_2, \ldots, Q_n$ and transform to new units; we choose the scales $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that $Q'_1, Q'_2, \ldots, Q'_n$ each has the value 1 in the new system of units. Imposing this condition on (2.2), we obtain

$$\alpha_1^{a_{j1}} \alpha_2^{a_{j2}} \ldots \alpha_n^{a_{jn}} = \frac{1}{Q_j}, \tag{2.5}$$

Solving these equations, we find that the α_j are given by

$$\alpha^j = \frac{1}{Q_1^{b_{j1}} Q_2^{b_{j2}} \ldots Q_n^{b_{jn}}}, \tag{2.6}$$

where b_{jk} are constants whose values we shall not determine. (We might suggest that the reader solve equations (2.5) for the simplest case of $n = 2$ if he is interested in the details of the derivation of (2.6)).

Let us now write (2.2) for the particular case where $j = 0$:

$$Q'_0 = \alpha_1^{a_{01}} \alpha_2^{a_{02}} \ldots \alpha_n^{a_{0n}} Q_0.$$

Substituting in this the value of α_j from (2.6), we obtain

$$Q_0' = \frac{Q_0}{Q_1^{x_1} Q_2^{x_2} \dots Q_n^{x_n}}, \tag{2.7}$$

where for the moment we do not determine the exponents $x_1, x_2, \dots, x_n$.

Inequality (2.3) does not depend on the units of measurement, and it is therefore also satisfied after going over to the units $q_1', q_2', \dots, q_n'$; thus

$$Q_0' \leqslant f(1, 1, \dots, 1).$$

Setting

$$C = f(1, 1, \dots, 1)$$

and substituting in the last inequality the expression for Q_0' given by (2.7), we obtain

$$\frac{Q_0}{Q_1^{x_1} Q_2^{x_2} \dots Q_n^{x_n}} \leqslant C,$$

and hence

$$Q_0 \leqslant C Q_1^{x_1} Q_2^{x_2} \dots Q_n^{x_n}.$$

Inequality (2.4) has thus been obtained. Now, according to condition (4), the x_i are such that this inequality is *dimensionally homogeneous*, i.e., the powers of the dimensions on the right and left-hand sides must be the same for each basic unit. Thus equating them, we obtain equations from which the x_i may be determined.

The theorem is proved.

3. Let us consider a system of n gravitating bodies $P_0, P_1, \dots, P_{n-1}$ with masses $m_0, m_1, \dots, m_{n-1}$. Let $r_{ij}(t)$ denote the distance from P_j to P_i and $r_{ij}'(t)$ the derivative

of this quantity with respect to time. We shall call the latter the *radial* velocity.

Let us introduce the notation

$$\rho(t)=\min_{ij}\{r_{ij}(t)\},$$

$$\sigma(t)=\min_{ij}\{r'_{ij}(t)\}.$$

We shall assume that at the initial time $t=0$ the condition

$$\sigma(0)>0. \tag{2.8}$$

is satisfied.

Under this assumption, there are two possibilities:

(a) it is possible to give a positive number τ such that, for all t in the half-open interval $[0,\tau)$, the distances between the bodies increase and from the moment of time τ at least one of the distances ceases to increase;

(b) or if $\sigma(0)$ is so large, and therefore also the initial values $r'_{ij}(0)$ of all the radial velocities, that the mutual attraction of the bodies of the system is not great enough to contain the system, then all the $r_{ij}(t)$ will approach infinity as $t\to\infty$.

However, for exactly what value of $\sigma(0)$ will this phenomenon occur? This, of course, will depend on the initial state of the system, on the parameters of the system and on *the parameters which characterize the phenomenon of gravitation in general.*

Let us try to clarify exactly which parameters are of great importance in the problem under consideration.

From general physical notions about the force of gravity, it follows that the smaller the value of $\rho(0)$, i.e., the closer the bodies are at the initial time, the greater must be the value of $\sigma(0)$; therefore, there is no doubt that one of the parameters is $\rho(0)$.

The masses $m_0, m_1, \ldots, m_{n-1}$ are also of considerable importance. For, the larger the masses of $P_0, P_1, \ldots, P_{n-1}$, the stronger will be their mutual attraction and, with all other conditions being equal, the greater must be the value of $\sigma(0)$. Thus, taking into account the masses will yield n more parameters. However, we can avoid the large number of parameters entailed by the masses if we remember that we are going to obtain sufficient conditions in the form of certain inequalities. Therefore for the purposes of simplifying the problem, we can use an overestimate of the role of the masses. As a parameter characterizing mass effects, we take

$$\mu = \varphi(m_0\, m_1, \ldots, m_{n-1}),$$

which is some function of the masses *having the dimension of mass*.

Finally, gravity is of essential importance, and therefore the gravitational constant γ must also be one of the parameters of our problem.

The parameters γ, μ, $\rho(0)$ and $\sigma(0)$ constitute a *complete* set of parameters for the phenomenon under consideration. In fact, for given values of $\rho(0)$ and μ, if $\sigma(0)$ is sufficiently great, then the bodies of the system will all be scattered, regardless of what values the other unaccounted for parameters may have. Thus we conclude that the required criterion must have the following general form:

$$\sigma(0) > f(\gamma, \mu, \rho(0)), \tag{2.9}$$

where the function f is momentarily unknown. We must now determine the functional form of $f(\gamma, \mu, \rho(0))$.

Consider arbitrary but fixed values of the parameters γ, μ, and $\rho(0)$; let $\sigma_*(0)$ denote the greatest lower bound

of the set of values of $\sigma(0)$ for which the system of bodies is completely scattered. This lower bound obviously exists because of inequality (2.8). It is further obvious that the inequality

$$\sigma_*(0) \leqslant f(\gamma, \mu, \rho(0)) \tag{2.10}$$

is satisfied for *all values of* γ, μ, *and* $\rho(0)$.

Let us take the system of units ordinarily used in problems in mechanics in which the basic units are time T, length L, and mass M. The parameters in which we are interested have the following dimensions:

$$\begin{aligned} [\rho(0)] &= L, \\ [\sigma_*(0)] &= LT^{-1}, \\ [\mu] &= M, \\ [\gamma] &= L^3 T^{-2} M^{-1}. \end{aligned} \tag{2.11}$$

We thus have four parameters, three basic units, and the inequality (2.10), i.e., our conditions correspond to those of Theorem 2.1, and to apply it, we must set $n=3$. By this theorem, we then have

$$\sigma_*(0) \leqslant C\gamma^x \mu^y \rho(0)^z, \tag{2.12}$$

where the exponents x, y, and z are for the moment unknown. To determine them, we equate the dimensions of the left-hand and right-hand sides of this inequality. This yields

$$LT^{-1} = T^{-2x} L^{3x} M^{-x} M^y L^z,$$

or

$$LT^{-1} = L^{3x+z} T^{-2x} M^{y-x}.$$

Equating exponents of like dimensions, we obtain the equations

$$2x=1,$$
$$y-x=0,$$
$$3x+z=1,$$

whose solution is

$$x=\frac{1}{2},\ y=\frac{1}{2},\ z=-\frac{1}{2}.$$

Substituting these values of x, y, and z in inequality (2.12), we obtain

$$\sigma_*(0)\leqslant C\sqrt{\frac{\gamma\mu}{\rho(0)}}. \tag{2.13}$$

However, if our system of units is the one usually used in theoretical investigations and in which $\gamma=1$, then the criterion becomes

$$\sigma_*(0)\leqslant C\sqrt{\frac{\mu}{\rho(0)}}. \tag{2.13a}$$

Our above result implies that

$$f(\gamma,\ \mu,\ \rho(0))=C\sqrt{\frac{\gamma\mu}{\rho(0)}},$$

and thus (2.9) becomes

$$\sigma(0)>C\sqrt{\frac{\mu}{\rho(0)}}\quad(\gamma=1). \tag{2.14}$$

The value of C and the form of the function expressing μ in terms of $m_0, m_1, \ldots$, and m_{n-1} cannot be determined by means of dimensional analysis. Thus dimensional analysis does not yield definitive sufficient conditons for the occurrence of the type of motions of n gravitating bodies in which we are interested; it merely determines the analytical form of the criterion. Later on, we shall see that the results obtained are confirmed by the direct analysis of the equations of motion.

In conclusion, let us examine the methodological aspects of our above investigations.

We obtained inequality (2.9) on the basis of general physical considerations. This part of the investigation does not concern dimensional analysis. Figuratively speaking, inequality (2.9) is not within the "competence" of dimensional analysis, and it cannot be "responsible" for this inequality. What merely follows from dimensional analysis is that if (2.9) is in principle correct, then it has the form (2.14). The question as to whether or not inequality (2.9) is the correct form of a sufficient condition for the occurrence of different types of motion cannot in principle be answered by dimensional analysis.

In this connection, it is interesting to note the following. Our reasoning cannot be applied to the case where $\sigma(0)<0$. In fact, if we wrote the inequality (2.9) in the form

$$|\sigma(0)|>f(\gamma, \mu, \rho(0)), \tag{2.15}$$

we could, at first glance, repeat our argument and obtain a similar but no more general result. However, in reality, this is not so. For, in assuming that $\sigma(0)<0$, we would not have the right to suppose that the problem is solved by inequalities (2.9) and (2.14). When $\sigma(0)<0$, it is possible for the gravitating bodies to approach one another

very closely, and the resulting gravitational interactions could then lead to such an exchange of energy between the bodies that a portion of them could go off to infinity with increased velocity, whereas the energy of the remaining bodies would be insufficient for them to be scattered. In this case, it is not sufficient to consider merely the parameters $\sigma(0)$, $\rho(0)$, μ and γ in formulating conditions for the scattering of gravitating bodies, and we require a more detailed set of parameters that characterizes more fully the initial state of the system of gravitating bodies. Therefore, when $\sigma(0)<0$, the relation (2.14) must either be replaced by one containing a fuller set of parameters, or it must be supplemented by relations containing other necessary parameters, or else our considerations must be limited to such a set of initial states for which the undesirable close approaches cannot occur.

Let v_{ij} be the magnitude of the velocity in the relative motion of P_i and P_j, and let

$$v(t)=\min_{ij}\{v_{ij}(t)\}.$$

It is clear that if motions of gravitating bodies are to occur for which $\rho(t)\to\infty$ as $t\to\infty$, it is not at all necessary for the radial velocities to be sufficiently great. Such motions can occur if the absolute velocities in the relative motion of the bodies are sufficiently great. Nevertheless, we cannot substitute $v(0)$ for $\sigma(0)$ in inequality (2.9), even though the dimensions of these quantities are identical. This is due to the fact $v(0)$ is always positive, whereas it is possible for $v(0)=|\sigma(0)|$ when $\sigma(0)<0$. But we saw above that a criterion of the form (2.9) does not apply to this particular case. Therefore, any criterion for the occurrence of motions in which $r_{ij}(t)\to\infty$ as $t\to\infty$ cannot contain merely $v(0)$, $\rho(0)$, μ, and γ, but must contain

still other parameters that, to one extent or another, explicitly or implicitly account for the direction of the velocity. As we shall see later on, this may be achieved by using $v(0)$ and $\sigma(0)$ together since their difference, to a certain extent, characterizes the direction of the velocity. In this way, one may obtain a criterion which is applicable for a sufficiently wide range of values of $\sigma(0) < 0$.

We have limited ourselves to a comparatively simple application of the method of dimensional analysis. However, this application makes rather well-apparent its characteristic features and the possibilities of its use.

In summing up, we must admit that the method of dimensional analysis requires great care and thought, as well as a deep and penetrating intuition into the essence of a problem. Nevertheless, dimensional analysis can be earnestly recommended if it is simply regarded as an exploratory or research means in theoretical considerations.

CHAPTER 3

Method of Continuous Induction

1. We shall consider a set $\mathfrak{M}$ whose elements are time-dependent processes or mechanical or physical systems S whose states vary with the time t.

Let Γ be some property of S. In what follows, we shall make use of the following definitions.

(1) The property Γ is said to be *inductive on the set* $\mathfrak{M}$ if for every $S \in \mathfrak{M}$ we have the following: from the assumption that the property Γ is satisfied for all values of t such that

$$0 \leqslant t < \tau,$$

it follows that it is satisfied for $t = \tau$ no matter what τ is.

(2) The property Γ is said to be *continuous* if from the condition that it is satisfied at a certain moment of time t_1, it follows that it is satisfied in a certain neighborhood of t_1, i.e., it is possible to find a number $\delta > 0$ such that Γ is satisfied for all t such that

$$|t - t_1| < \delta.$$

THEOREM 3.1. (On Continuous Induction). *Let the property* Γ *be continuous and inductive on the set* $\mathfrak{M}$. *If* Γ *is satisfied at the initial time* $t = 0$, *then it is satisfied for all values of* $t > 0$.

Proof: We call the time t *regular* if for all t' such that

$$0 \leqslant t' \leqslant t,$$

the system S possesses the property Γ. Let Q denote the set of all regular times. From the continuity of Γ and the assumption that Γ is satisfied at $t = 0$, it follows that the set Q is not empty.

We prove the theorem by contradiction, and we assume that Γ is not satisfied for all values of $t \geqslant 0$. Therefore, Q is bounded from above; let $\tau = \sup Q$. Since τ is the least upper bound, it follows that τ is satisfied for all values of t in the interval $0 \leqslant t < \tau$. But then since Γ is inductive, it is satisfied at time τ. Furthermore, the property Γ is continuous. Therefore, we can find a $\delta > 0$ such that Γ is satisfied for all t for which

$$|t-\tau| < \delta,$$

But then Γ must be satisfied for values of t for which

$$0 \leqslant t \leqslant \tau + \frac{\delta}{2}.$$

Hence, it follows that the time

$$\tau_1 = \tau + \frac{\delta}{2} > \tau$$

is regular. However, this contradicts the fact that τ is the least upper bound of the set Q. The theorem is proved.

2. To apply the Theorem on Continuous Induction, we must have criteria for when some particular property is actually inductive. We shall now show that such criteria can be given in certain sufficiently general cases.

Let

$$\omega_1, \omega_2, \ldots, \omega_\nu$$

be some set of parameters of the system S depending on the time which, while not completely determining the state of the system, characterizes it in certain respects. We denote the set of functions

$$\omega_1, \omega_2, \ldots, \omega_\nu$$

by $\{\omega\}$. Moreover, let the system depend on μ parameters

$$a_1,\ a_2,\ \ldots,\ a_\mu,$$

which are independent of the time, and which we denote collectively by $\{a\}$.

Let $A(\{a\},\{\omega\})$ denote an operator which, according to certain rules, sets in correspondence to each set of values of the parameters $a_1, a_2, \ldots, a_\mu$ and to each set of ν functions $\omega_1, \omega_2, \ldots, \omega_\nu$ a continuous function $\Psi(t)$ defined for all $t \geqslant 0$.

We first consider the case where $\omega_1, \omega_2, \ldots, \omega_\nu$ are unknown functions, and our knowledge of the system S is restricted to the information that these functions and the parameters $a_1, a_2, \ldots, a_\mu$ satisfy relations of the form

$$\underset{i=1,\ 2,\ldots,\ \nu}{\omega_i(t) \geqslant A_i(\{a\},\{\omega\})}. \tag{3.1}$$

Here, A_i are operators which are defined by means of their analytical expressions.

Let

$$f_1(t), f_2(t), \ldots, f_\nu(t)$$

be suitably-chosen continuous functions of t defined for all $t \geqslant 0$ which we shall call *comparison* functions. The property Γ will be that the following inequalities are satisfied:

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) > f_i(t)}.$$

THEOREM 3.2 (First Theorem on the Inductiveness of Inequalities). *If a system S satisfies the operator inequalities* (3.1), *and if for any* $\tau \geqslant 0$, *from the assumption that the inequalities*

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) > f_i(t)} \tag{3.2}$$

We prove the theorem by contradiction, and we assume that Γ is not satisfied for all values of $t \geqslant 0$. Therefore, Q is bounded from above; let $\tau = \sup Q$. Since τ is the least upper bound, it follows that τ is satisfied for all values of t in the interval $0 \leqslant t < \tau$. But then since Γ is inductive, it is satisfied at time τ. Furthermore, the property Γ is continuous. Therefore, we can find a $\delta > 0$ such that Γ is satisfied for all t for which

$$|t - \tau| < \delta,$$

But then Γ must be satisfied for values of t for which

$$0 \leqslant t \leqslant \tau + \frac{\delta}{2}.$$

Hence, it follows that the time

$$\tau_1 = \tau + \frac{\delta}{2} > \tau$$

is regular. However, this contradicts the fact that τ is the least upper bound of the set Q. The theorem is proved.

2. To apply the Theorem on Continuous Induction, we must have criteria for when some particular property is actually inductive. We shall now show that such criteria can be given in certain sufficiently general cases.

Let

$$\omega_1, \omega_2, \ldots, \omega_\nu$$

be some set of parameters of the system S depending on the time which, while not completely determining the state of the system, characterizes it in certain respects. We denote the set of functions

$$\omega_1, \omega_2, \ldots, \omega_\nu$$

by $\{\omega\}$. Moreover, let the system depend on μ parameters

$$a_1, a_2, \ldots, a_\mu,$$

which are independent of the time, and which we denote collectively by $\{a\}$.

Let $A(\{a\},\{\omega\})$ denote an operator which, according to certain rules, sets in correspondence to each set of values of the parameters $a_1, a_2, \ldots, a_\mu$ and to each set of ν functions $\omega_1, \omega_2, \ldots, \omega_\nu$ a continuous function $\Psi(t)$ defined for all $t \geqslant 0$.

We first consider the case where $\omega_1, \omega_2, \ldots, \omega_\nu$ are unknown functions, and our knowledge of the system S is restricted to the information that these functions and the parameters $a_1, a_2, \ldots, a_\mu$ satisfy relations of the form

$$\underset{i=1,\ 2,\ldots,\ \nu}{\omega_i(t) \geqslant A_i(\{a\}, \{\omega\}).} \tag{3.1}$$

Here, A_i are operators which are defined by means of their analytical expressions.

Let

$$f_1(t), f_2(t), \ldots, f_\nu(t)$$

be suitably-chosen continuous functions of t defined for all $t \geqslant 0$ which we shall call *comparison* functions. The property Γ will be that the following inequalities are satisfied:

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) > f_i(t).}$$

THEOREM 3.2 (First Theorem on the Inductiveness of Inequalities). *If a system S satisfies the operator inequalities* (3.1), *and if for any $\tau \geqslant 0$, from the assumption that the inequalities*

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) > f_i(t)} \tag{3.2}$$

are satisfied in the half-closed interval $0 \leqslant t < \tau$, *it follows that the inequalities*

$$A_i(\{a\}), \{\omega\}) \geqslant A_i(\{a\}, \{f\}), \tag{3.3}$$

are satisfied, then (3.2) *is inductive on the set* $\mathfrak{M}$ *of systems S for which the inequalities*

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{A_i(\{a\}, \{f\}) - f_i(t) > 0.} \tag{3.4}$$

are satisfied for all $t \geqslant 0$.

Proof: Let us write inequality (3.1) in the form

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) - f_i(t) \geqslant A_i(\{a\}, \{\omega\}) - f_i(t).} \tag{3.5}$$

Choosing an arbitrary $\tau > 0$, we assume that

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) > f_i(t) \text{ for } 0 \leqslant t < \tau.} \tag{3.6}$$

Then considering (3.5) for values of t in the interval $0 \leqslant t < \tau$ and using (3.3), which holds because of (3.6), we can write

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(t) - f_i(t) \geqslant A_i(\{a\}, \{f\}) - f_i(t).}$$

We now let $t \to \tau$ taking into account (3.4) and the fact that $\omega_i(t)$, $f_i(t)$, and $A_i(\{a\}, \{\omega\})$ are continuous functions of t, and we obtain in the limit

$$\underset{i=1,\ 2,\ \ldots,\ \nu}{\omega_i(\tau) - f_i(\tau) \geqslant A_i(\{a\}, \{f\}) - f_i(\tau) > 0,}$$

or

$$\omega_i(\tau) > f_i(\tau). \qquad (3.7)$$
$$i=1,\ 2,\ \ldots,\ \nu$$

We have thus shown that the inequality (3.7) follows from (3.6), and this proves the theorem.

3. We now consider a more complicated case of the Theorem on the Inductiveness of Inequalities which we shall prove in two parallel versions.

Suppose we have two sets of parameters of the system S which depend on time:

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_\nu$$
$$\delta_1,\ \delta_2,\ \ldots,\ \delta_\nu$$

which, though not defining the state of the system completely, characterize it in certain respects. As before, suppose the state of the system also depends on μ parameters $a_1, a_2, \ldots, a_\mu$ which are independent of time. We consider the case in which $\omega_1, \omega_2, \ldots, \omega_\nu$ and $\delta_1, \delta_2, \ldots, \delta_\nu$ are unknown functions, and our knowledge of the system S is restricted to the fact that these functions and the parameters $a_1, a_2, \ldots, a_\mu$ satisfy relations of the form

$$\begin{gathered} \delta_i(t) \leqslant A_i(\{a\},\ \{\omega\}) \\ \big(\delta_i(t) \geqslant A_i(\{a\},\ \{\omega\})\big). \\ i=1,\ 2,\ \ldots,\ \nu \end{gathered} \qquad (3.8)$$

Here, A_i are operators defined by their analytic expressions.

Let

$$\varphi_1(t),\ \varphi_2(t),\ \ldots,\ \varphi_\nu(t),$$
$$f_1(t), f_2(t),\ \ldots,\ f_\nu(t)$$

be suitably chosen continuous functions of the time t

which are defined on the range $t \geqslant 0$, and which we call comparison functions. We shall assume that they possess the following property: From the assumption that the inequalities

$$\begin{gathered} \delta_i(t) \leqslant \varphi_i(t) \\ \left(\delta_i(t) \geqslant \varphi_i(t)\right), \\ {\scriptstyle i=1,2,\ldots,\nu} \end{gathered} \tag{3.9}$$

are satisfied in the interval $0 \leqslant t < \tau$, it follows that the inequalities

$$\begin{gathered} \omega_i(t) \geqslant f_i(t). \\ {\scriptstyle i=1,2,\ldots,\nu} \end{gathered} \tag{3.10}$$

are satisfied for all values of t.

Although we do not know the functions $\omega_i(t)$ and $\delta_i(t)$, we can, however, often establish such relations from the physical or mechanical meaning of these functions or from other considerations not requiring the knowledge of the analytical form of these functions.

We shall take Γ to be the property that the inequalities (3.9) are satisfied.

THEOREM 3.3 (Second Theorem on the Inductiveness of Inequalities). *If the system S satisfies the operator inequalities* (3.8), *and if for any $\tau > 0$, from the assumption that the inequalities* (3.10) *are satisfied in the interval $0 \leqslant t < \tau$ it follows that the inequalities*

$$\begin{gathered} A_i(\{a\},\ \{\omega\}) \leqslant A_i(\{a\},\ \{f\}) \\ \left(A_i(\{a\},\ \{\omega\}) \geqslant A_i(\{a\},\ \{f\})\right), \\ {\scriptstyle i=1,2,\ldots,\nu} \end{gathered} \tag{3.11}$$

are satisfied, then property (3.9) (*and* (3.10) *which follows from it*) *is inductive on the set* $\mathfrak{M}$ *of systems S for which the inequalities*

$$\begin{gathered} A_i(\{a\},\ \{f\}) - \varphi_i(t) < 0 \\ \left(A_i(\{a\}, \{f\}) - \varphi_i(t) > 0\right). \\ i=1,2,\ldots \end{gathered} \tag{3.12}$$

are satisfied for all $t \geqslant 0$.

Proof: Let us write inequality (3.8) in the form

$$\begin{gathered} \delta_i(t) - \varphi_i(t) \leqslant A_i(\{a\},\ \{\omega\}) - \varphi_i(t) \\ \left(\delta_i(t) - \varphi_i(t) \geqslant A_i(\{a\},\ \{\omega\}) - \varphi_i(t)\right) \\ i=1,2,\ldots \end{gathered} \tag{3.13}$$

Choosing an arbitrary $\tau > 0$, we assume that

$$\begin{gathered} \delta_i(t) < \varphi_i(t), \\ 0 \leqslant t < \tau \\ \left(\delta_i(t) > \varphi_i(t)\right). \\ i=1,2,\ldots, \end{gathered} \tag{3.14}$$

Then for these same values of t, the inequalities

$$\begin{gathered} \omega_i(t) > f_i(t). \\ i=1,2,\ldots \end{gathered} \tag{3.15}$$

will be satisfied. Considering (3.13) for values of t in the interval $0 \leqslant t < \tau$ and making use of (3.11), which hold because of (3.15), we can write

We now let $t \to \tau$ taking into account (3.12) and the fact that $\omega_i(t)$, $\delta_i(t)$, $f_i(t)$, $\varphi_i(t)$, and $A_i(\{a\},\{\omega\})$ are continuous functions of t, and we obtain in the limit

$$\delta_i(\tau) - \varphi_i(\tau) \leqslant A_i(\{a\},\ \{f\}) - \varphi_i(\tau) < 0$$
$$\left(\delta_i(\tau) - \varphi_i(\tau) \geqslant A_i(\{a\},\ \{f\}) - \varphi_i(\tau) > 0\right),$$
$$i = 1, 2, \ldots,$$

or

$$\delta_i(\tau) < \varphi_i(\tau)$$
$$\left(\delta_i(\tau) > \varphi_i(\tau)\right), \tag{3.16}$$
$$i = 1, 2, \ldots, \nu$$

From this, in turn, it follows that

$$\omega_i(\tau) > f_i(\tau).$$
$$i = 1, 2, \ldots, \nu$$

We have thus shown that inequality (3.16) follows from (3.14) with which the theorem is proved.

4. We now consider some applications of the Theorems on Continuous Induction and the Inductiveness of Inequalities to the qualitative analysis of gravitating bodies.

We begin with the first Theorem on the Inductiveness of Inequalities.

Let us take a system of n gravitating bodies $P_0, P_1, \ldots, P_{n-1}$ with masses $m_0, m_1, \ldots, m_{n-1}$, and let

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j}.$$

We shall make use of Jacobi coordinates. To begin with,

we consider the motion of P_1 relative to P_0. Differentiating the identity

$$r_{01}^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2,$$

twice with respect to t, we obtain

$$r_{01} r_{01}'' + r_{01}'^2 = \xi_1 \xi_1'' + \eta_1 \eta_1'' + \zeta_1 \zeta_1'' + \xi_1'^2 + \eta_1'^2 + \zeta_1'^2,$$

or

$$r_{01}'' = \frac{\xi_1}{r_{01}} \xi_1'' + \frac{\eta_1}{r_{01}} \eta_1'' + \frac{\zeta_1}{r_{01}} \zeta_1'' + \\ + \frac{\xi_1'^2 + \eta_1'^2 + \zeta_1'^2 - r_{01}'^2}{r_{01}}.$$

Taking into account that

$$\frac{\xi_1'^2 + \eta_1'^2 + \zeta_1'^2 - r_{01}'^2}{r_{01}} \geqslant 0,$$

we find that

$$r_{01}'' \geqslant \frac{\xi_1}{r_{01}} \xi_1'' + \frac{\eta_1}{r_{01}} \eta_1'' + \frac{\zeta_1}{r_{01}} \zeta_1''.$$

Using the first of equations (1.7), i.e., for $i=1$, we obtain

$$r_{01}'' \geqslant \frac{1}{\mu_{01}} \left(\frac{\xi_1}{r_{01}} \frac{\partial U}{\partial \xi_1} + \frac{\eta_1}{r_{01}} \frac{\partial U}{\partial \eta_1} + \frac{\zeta_1}{r_{01}} \frac{\partial U}{\partial \zeta_1} \right).$$

Now, let l denote the direction of the line joining P_0 to P_1. Then the last inequality may be written in the following form

$$r''_{01} \geqslant \frac{1}{\mu_{01}} \frac{\partial U}{\partial l} \, . \tag{3.17}$$

Differentiating the potential function U in the direction l using (1.1), we have

$$\frac{\partial U}{\partial l} = -\sum_{ij} \frac{m_i m_j}{r_{ij}^2} \frac{\partial r_{ij}}{\partial l} \, .$$

From simple geometrical considerations, it follows that

$$\left| \frac{\partial r_{ij}}{\partial l} \right| \leqslant 1,$$

and, therefore,

$$\frac{\partial U}{\partial l} \geqslant -\sum_{ij} \frac{m_i m_j}{r_{ij}^2} \, . \tag{3.18}$$

Inequalities (3.17) and (3.18) imply that

$$r''_{01} \geqslant -\frac{1}{\mu_{01}} \sum_{ij} \frac{m_i m_j}{r_{ij}^2} \, .$$

We have obtained this inequality by considering how P_1 moves relative to P_0. However, we could have considered the motion of any point P_s relative to any other point P_k. Thus, we have

$$r''_{ks} \geqslant -\frac{1}{\mu_{ks}} \sum_{ij} \frac{m_i m_j}{r_{ij}^2} \, .$$

Finally, integrating this inequality twice with respect to time from $t=0$ to $t>0$, we obtain the following system of integral inequalities:

$$r_{ks}(t) \geqslant r_{ks}(0) + r'_{ks}(0)\,t - \frac{1}{\mu_{ks}} \sum_{ij} \int_0^t \int_0^t \frac{m_i m_j}{r_{ij}^2}\,dt\,dt. \quad (3.19)$$

We now use these inequalities to find sufficient conditions which ensure that the distance of each body from the remaining ones will become infinite as $t \to \infty$, in other words, that the system is completely scattered. Such will be the case, at any rate, if the inequalities

$$r_{ks}(t) > \frac{r_{ks}(0)}{2} + \frac{r'_{ks}(0)}{2}\,t$$

are satisfied for all $t \geqslant 0$.

To find the conditions, it is most convenient to use the first Theorem on the Inductiveness of Inequalities (Theorem 3.2).

For the functions $\omega_i(t)$, we take the distances $r_{ks}(t)$ and for the parameters a_i, the initial values of these distances and their derivatives, $r_{ks}(0)$ and $r'_{ks}(0)$, and the masses $m_0, m_1, \ldots, m_{n-1}$.

In accordance with inequality (*), it is natural to choose the comparison functions to be

$$f_{ks}(t) = \frac{r_{ks}(0)}{2} + \frac{r'_{ks}(0)}{2}\,t.$$

We define the operators A_i, using (3.19) as follows:

$$A_{ks} = r_{ks}(0) + r'_{ks}(0)\,t - \frac{1}{\mu_{ks}} \sum_{ij} \int_0^t \int_0^t \frac{m_i m_j}{r_{ij}^2}\,dt\,dt. \quad (3.20)$$

Finally we take Γ to be the property that the following inequalities are satisfied:

$$r_{ks}(t) > f_{ks}(t). \tag{3.21}$$

The conditions of Theorem 3.2 are satisfied since, from the assumption that (3.21) are satisfied in the interval $0 \leqslant t < \tau$, it follows that

$$-\frac{1}{\mu_{ks}} \sum_{ij} \int_0^t \int_0^t \frac{m_i m_j}{r_{ij}^2}\, dt\, dt >$$

$$> -\frac{1}{\mu_{ks}} \sum_{ij} \int_0^t \int_0^t \frac{m_i m_j}{f_{ij}^2}\, dt\, dt$$

Therefore, by Theorem 3.2, Γ is an inductive property for those sytems of gravitating bodies for which the inequalities

$$\frac{r_{ks}(0)}{2} + \frac{r_{ks}(0)}{2} t -$$

$$-\frac{1}{\mu_{ks}} \sum_{ij} \int_0^t \int_0^t \frac{m_i m_j}{\left(\frac{r_{ij}(0)}{2} + \frac{r'_{ij}(0)}{2} t\right)^2}\, dt\, dt > 0. \tag{3.22}$$

are satisfied for all $t \geqslant 0$. These inequalities should be considered as defining the appropriate values of $r_{ij}(0)$ and $r'_{ij}(0)$.

Evaluating the integral on the left-hand side of (3.22) we find

$$\int_0^t \int_0^t \frac{m_i m_j}{\left(\frac{r_{ij}(0)}{2} + \frac{r'_{ij}(0)}{2} t\right)^2} \, dt \, dt =$$

$$= \frac{4 m_i m_j}{r_{ij}(0) \, r'_{ij}(0)} t - \frac{4 m_i m_j}{r'^2_{ij}(0)} \ln \left(1 + \frac{r'_{ij}(0)}{r_{ij}(0)} t\right).$$

Substituting the value of the integral in (3.22), we obtain after simplification

$$\left\{\frac{r'_{ks}(0)}{2} - \frac{1}{\mu_{ks}} \sum_{ij} \frac{4 m_i m_j}{r_{ij}(0) \, r'_{ij}(0)}\right\} t + \\ + \frac{r_{ks}(0)}{2} + \frac{1}{\mu_{ks}} \sum_{ij} \frac{4 m_i m_j}{r'^2_{ij}(0)} \ln \left(1 + \frac{r'_{ij}(0)}{r_{ij}(0)} t\right) > 0. \tag{3.23}$$

The second and third terms on the left-hand side are positive for every value of t > 0, provided that $r'_{ij}(0) > 0$. Thus, (3.23) will be satisfied for all positive t if the coefficient of t in the first term on the left-hand side is positive, i.e., if

$$\frac{r'_{ks}(0)}{2} - \frac{1}{\mu_{ks}} \sum_{ij} \frac{4 m_i m_j}{r_{ij}(0) \, r'_{ij}(0)} > 0.$$

When these conditions are satisfied, inequality (3.21) will hold for all positive t, provided (3.21) holds for $t=0$. But then (3.21) is satisfied at $t=0$ since

$$f_{ks}(0) = \frac{r_{ks}(0)}{2}.$$

We can thus formulate the following theorem.

THEOREM 3.4. *If the conditions*

$$r'_{ij}(0) > 0,$$

$$\frac{r'_{ij}(0)}{2} - \frac{1}{\mu_{ks}} \sum_{ij} \frac{4m_i m_j}{r_{ij}(0)\, r'_{ij}(0)} > 0, \tag{3.24}$$

are satisfied at the initial time $t=0$ for a system of gravitating bodies, then the distances r_{ij} between the bodies tend to infinity as $t \to \infty$.

5. The criterion (3.24) just obtained is rather simple, but, nevertheless, slightly cumbersome. However, by weakening it, we can reduce the criterion to a simpler and more transparent form.

Let us introduce the notation

$$\rho(t) = \min_{ij} \{r_{ij}(t)\},$$

$$\sigma(t) = \min_{ij} \left\{ \frac{dr_{ij}(t)}{dt} \right\},$$

$$M^* = \frac{\sum_{ij} m_i m_j}{\min \mu_{ij}}.$$

It is not difficult to see that conditions (3.24) hold if

$$\sigma(0) > 0,$$

$$\sigma^2(0) > \frac{8M^*}{\rho(0)}. \tag{3.25}$$

Thus, if conditions (3.25) are satisfied, the distances r_{ij} between the bodies tend to infinity as $t \to \infty$.

We might note that conditions (3.25) have exactly the same analytic form as those which were obtained in Chapter 2 using dimensional analysis.

6. Let us now consider a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$, the motion of which we shall describe using Jacobi coordinates. Set $\rho = \rho_{n-1}$, where ρ_{n-1} denotes the distance of P_{n-1} from the center of mass of the remaining bodies. Choose a positive number $\alpha \leqslant 1$, and consider the sphere $S(\alpha)$ with center at the point P_{n-1} and with radius $R = \alpha\rho$. Thus, the center of the sphere moves together with the point P_{n-1}, and its radius varies with time. We shall say that an *α-approach* occurs between the body P_{n-1} and some other body P_i at time t if at this time P_i lies inside or on the boundary of the sphere $S(\alpha)$.

Differentiating the identity

$$\rho^2 = \xi_{n-1}^2 + \eta_{n-1}^2 + \zeta_{n-1}^2$$

twice with respect to the time, and carrying out the same kind of manipulations as was done for r_{01} in Section 4, we obtain

$$\rho'' \geqslant \frac{1}{\mu_{n-1}} \cdot \frac{\partial U}{\partial l}, \tag{3.26}$$

where l is the direction of the line joining the center of mass of the bodies $P_0, P_1, \ldots, P_{n-2}$ to the point P_{n-1}. Since in differentiation with respect to the direction l only the distances between P_{n-1} and the remaining bodies vary, we have that

$$\frac{\partial U}{\partial l} = -\sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \cdot \frac{\partial r_{i,n-1}}{\partial l}.$$

From simple geometrical considerations, it follows that

$$\left|\frac{\partial r_{i,n-1}}{\partial l}\right| \leqslant 1.$$

Therefore

$$\frac{\partial U}{\partial l} \geqslant -\sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2}. \tag{3.27}$$

Inequalities (3.26) and (3.27) thus imply that

$$\rho'' \geqslant -\frac{1}{\mu_{n-1}} \sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2}.$$

If it is assumed that no α-approach between P_{n-1} and the remaining bodies of the system occurs when $t \geqslant 0$, then

$$r_{i,n-1} > \alpha\rho.$$

Using this to strengthen the previous inequality, we obtain

$$\rho'' \geqslant -\frac{M}{\alpha^2\rho}, \tag{3.28}$$

where

$$M = \sum_{i=0}^{n-1} m_i.$$

Finally, if we integrate (3.28) twice with respect to time from $t=0$ to $t>0$, we obtain the integral inequality

$$\rho(t) > \rho(0) + \rho'(0)t - \frac{M}{\alpha^2}\int_0^t\int_0^t \frac{dt}{\rho^2}. \tag{3.29}$$

THEOREM 3.5. *If the initial conditions*

$$\rho'(0) > 0,$$
$$\rho'^2(0) - \frac{8M}{a\rho(0)} > 0, \tag{3.30}$$

hold at $t = 0$ for a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$ and if no a-approach of P_{n-1} to the other bodies occurs for any value of $t \geqslant 0$, then the distance of P_{n-1} from each of the other bodies in the system becomes infinite as $t \to \infty$.

Proof: The integral inequality (3.29) is of fhe same type as those in the system (3.19). Applying Theorem 3.2 to it and repeating the argument that was used in the proof of Theorem 3.4, we obtain a proof of the stated theorem.

7. We now consider some applications of the second Theorem on the Inductiveness of Inequalities. As a first example, it will be constructive to consider one of the cases already treated with the help of the first Theorem on the Inductiveness of Inequalities. Let us turn again to Theorem 3.5, and let us prove it in a slightly stronger form. In so doing, we obtain some results which we shall find useful below.

THEOREM 3.5a. *If the initial conditions*

$$\rho'(0) > 0,$$
$$\rho'^2(0) - \frac{4M}{a^2\rho(0)} > 0,$$

hold at $t = 0$ for a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$, and if no a-approach of P_{n-1} to the other bodies occurs for any value of $t \geqslant 0$, then the inequalities

$$\rho'(t) > \frac{\rho'(0)}{2}, \tag{3.31}$$

$$\rho(t) \geqslant \rho(0) + \frac{\rho'(0)}{2}t, \tag{3.32}$$

hold for all positive t, and, therefore, $\rho(t) \to \infty$ as $t \to \infty$. This implies that the distance of P_{n-1} from each of the other bodies in the system becomes infinite as $t \to \infty$.

To prove the theorem, we start with inequality (3.28). Integrating it with respect to time from $t=0$ to $t>0$, we obtain

$$\rho'(t) \geqslant \rho'(0) - \frac{M}{\alpha^2}\int_0^t \frac{dt}{\rho^2}.$$

Applying the parallel version of Theorem 3.3, we set

$$\delta(t) = \rho'(t),$$

$$\omega(t) = \rho(t),$$

and we define the operator A by means of the equation

$$A = \rho'(0) - \frac{M}{\alpha^2}\int_0^t \frac{dt}{\rho^2}.$$

For the comparison functions, we take

$$\varphi(t) = \frac{\rho'(0)}{2},$$

$$f(t) = \rho(0) + \frac{\rho'(0)}{2}t.$$

It is first necessary to show that the conditions of Theorem 3.3 are satisfied for the case in question. With

our choice of $\delta(t)$, $\omega(t)$, $\varphi(t)$, and $f(t)$, it is obvious that (3.9) holds. The operator A also satisfies the required conditions since from the asumption that $\rho(t)>f(t)$ in the interval $0\leqslant t<\tau$, it follows that the inequality

$$\rho'(0)-\frac{M}{\alpha^2}\int_0^t\frac{dt}{\rho^2}>\rho'(0)-\frac{M}{\alpha^2}\int_0^t\frac{dt}{f^2}.$$

is satisfied for these same values of t. Therefore, by Theorem 3.3, inequality (3.31), as well as inequality (3.32) which follows from it, are inductive properties for systems of gravitating bodies for which the inequality

$$\frac{\rho'(0)}{2}-\frac{M}{\alpha^2}\int_0^t\frac{dt}{\left(\rho(0)+\frac{\rho'(0)}{2}t\right)^2}>0,$$

holds for all values of $t\geqslant 0$. After integration, this last inequality becomes

$$\frac{\rho'^2(0)}{2}-\frac{2M}{\alpha^2\rho(0)}+\frac{2M}{\alpha^2\left(\rho(0)+\frac{\rho'(0)}{2}t\right)}>0.$$

It is easy to see that when the conditions of the theorem are satisfied, this inequality holds for all values of $t\geqslant 0$. Therefore, under the conditions of the theorem, (3.31) and (3.32) are inductive properties, and since they hold at the initial time $t=0$, they are also valid for all $t\geqslant 0$.

The theorem is proved.

8. Let us now consider some further applications of the second Theorem on the Inductiveness of Inequalities.

Suppose we have a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$ with masses $m_0, m_1, \ldots, m_{n-1}$. To de-

scribe their motion, we make use of equations (1.12). Integrating these equations with respect to time from $t=0$ to $t>0$, we obtain

$$\mathbf{r}'_{ik}-\mathbf{r}'_{ik}(0)=-m_k\int_0^t\frac{\mathbf{r}_{ik}}{r_{ik}^3}\,dt+\sum_{\substack{j=0\\ j\neq i,k}}^{n-1}m_j\int_0^t\frac{\mathbf{r}_{ij}}{r_{ij}^3}\,dt-$$
$$-\sum_{\substack{j=1\\ j\neq k}}^{n-1}m_j\int_0^t\frac{\mathbf{r}_{jk}}{r_{jk}^3}\,dt.$$

From this, it follows that

$$|\mathbf{r}'_{ik}-\mathbf{r}'_{ik}(0)|\leqslant m_k\int_0^t\frac{dt}{r_{ik}^2}+\sum_{\substack{j=0\\ j\neq i,k}}^{n-1}m_j\int_0^t\frac{dt}{r_{ij}^2}+$$
$$+\sum_{\substack{j=1\\ j\neq k}}^{n-1}m_j\int_0^t\frac{dt}{r_{jk}^2}. \tag{3.33}$$

With a view towards applying Theorem 3.3, we set

$$\omega_{ik}(t)=r_{ik}^2(t),$$
$$\delta_{ik}(t)=|\mathbf{r}'_{ik}-\mathbf{r}'_{ik}(0)|,$$

and we define the operators $A(\{a_{ik}\}),\{\omega\})$ by means of the equations

$$A_{ik}=m_k\int_0^t\frac{dt}{r_{ik}^2}+\sum_{\substack{j=0\\ j\neq i,k}}^{n-1}m_j\int_0^t\frac{dt}{r_{ij}^2}+\sum_{\substack{j=1\\ j\neq k}}^{n-1}m_j\int_0^t\frac{dt}{r_{jk}^2}. \tag{3.34}$$

We can thus consider (3.33) to be operator inequalities satisfied by the system of gravitating bodies.

We now assume the initial conditions are such that

$$v_{ik}(0) - 2r'_{ik}(0) > 0,$$
$$v^2_{ik}(0) - r'^2_{ik}(0) > 0, \tag{3.35}$$

and we define the comparison functions by means of the equations

$$\varphi_{ik}(t) = \frac{1}{3}\left[v_{ik}(0) + r'_{ik}(0)\right],$$

$$f_{ik}(t) = \left[\frac{2}{3} v_{ik}(0) - \frac{1}{3} r'_{ik}(0)\right]^2 t^2 +$$
$$+ 2r_{ik}(0)\left[\frac{2}{3} r'_{ik}(0) - \frac{1}{3} v_{ik}(0)\right] t + r^2_{ik}(0),$$

Finally, Γ will be considered to be the property that the inequalities

$$|\mathbf{r}'_{ik}(t) - \mathbf{r}'_{ik}(0)| < \frac{1}{3}\left[v_{ik}(0) + r'_{ik}(0)\right],$$

hold, where $v_{ik}(0) = |\mathbf{r}'_{ik}(0)|$.

To prove the inductiveness of Γ, we require the following lemma (G.A. Merman, 1953a).

LEMMA 3.1. *If the vector* $\mathbf{r}(t)$ *satisfies the inequality*

$$|\mathbf{r}' - \mathbf{r}|(0) < \epsilon$$

in the interval $0 \leqslant t < \tau$, *then it satisfies the inequality*

$$r^2 > [v(0) - \epsilon]^2 t^2 + 2r(0)[r'(0) - \epsilon] t + r^2(0),$$

for these same values of t, where $v(0)=|\mathbf{r}'(0)|$.

Proof: From the integral identity

$$\mathbf{r}=\mathbf{r}(0)+\mathbf{r}'(0)\,t+\int_0^t[\mathbf{r}'-\mathbf{r}'(0)]\,dt$$

we obtain

$$r\geqslant|\mathbf{r}(0)+\mathbf{r}'(0)\,t|-\left|\int_0^t[\mathbf{r}'-\mathbf{r}'(0)]\,dt\right|,$$

Hence, by the hypothesis of the lemma we have

$$\underset{0\leqslant t<\tau}{r>|\mathbf{r}(0)+\mathbf{r}'(0)\,t|-\varepsilon t,}$$

or

$$r^2>[\mathbf{r}(0)+\mathbf{r}'(0)\,t]^2-2\varepsilon\,|\mathbf{r}(0)+\mathbf{r}'(0)\,t|\,t+\varepsilon^2t^2,$$

which proves the lemma when the inequality

$$|\mathbf{r}(0)+\mathbf{r}'(0)\,t|\leqslant r(0)+v(0)\,t.$$

is applied.

Using this lemma, we now show that if the property Γ, i.e., the inequalities

$$|\mathbf{r}'_{ik}-\mathbf{r}'_{ik}(0)|<\frac{1}{3}[v_{ik}(0)+r'_{ik}(0)],\qquad(3.36)$$

hold in the interval $0\leqslant t<\tau$, then the inequalities

$$r_{ik}^2>\left|\frac{2}{3}v_{ik}(0)-\frac{1}{3}r'_{ik}(0)\right|^2t^2+$$

$$+2r_{ik}(0)\left[\frac{2}{3}r'_{ik}(0)-\frac{1}{3}v_{ik}(0)\right]t+r_{ik}^2(0).\qquad(3.37)$$

will also hold for these same values of t.

To prove this fact, we apply Lemma 3.1 for which it suffices to set

$$\mathbf{r} = \mathbf{r}_{ik},$$

$$\varepsilon = \frac{1}{3}\left[v_{ik}(0) + r'_{ik}(0)\right]$$

and to carry out some simple manipulations.

From the expression (3.34) which defines the operators A_{ij}, it immediately follows that if it is assumed that $r_{ik}^2(t) > f_{ik}(t)$ in the interval $0 \leqslant t < \tau$, then the inequalities

$$m_k \int_0^t \frac{dt}{r_{ik}^2} + \sum_{\substack{j=0 \\ j \neq i,k}}^{n-1} m_j \int_0^t \frac{dt}{r_{ij}^2} + \sum_{\substack{j=1 \\ j \neq k}}^{n-1} m_j \int_0^t \frac{dt}{r_{ik}^2} < m_k \int_0^t \frac{dt}{f_{ik}} +$$

$$+ \sum_{\substack{j=0 \\ j \neq i,k}}^{n-1} m_j \int_0^t \frac{dt}{f_{ij}} + \sum_{\substack{j=1 \\ j \neq k}}^{n-1} m_j \int_0^t \frac{dt}{f_{ik}}. \tag{3.38}$$

will hold for these same values of t.

The results obtained show that the conditions of the second Theorem on the Inductiveness of Inequalities are satisfied by the case under consideration. Therefore, the inequalities (3.36), as well as the inequalities (3.37) which follow from them, are inductive properties for systems of gravitating bodies which satisfy the inequalities

$$\begin{aligned} &m_k \int_0^t \frac{dt}{f_{ik}} + \sum_{\substack{j=0 \\ j \neq i,k}}^{n-1} m_j \int_0^t \frac{dt}{f_{ij}} + \\ &+ \sum_{\substack{j=1 \\ j \neq k}}^{n-1} m_j \int_0^t \frac{dt}{f_{jk}} - \frac{1}{3}\left[v_{ik}(0) + r_{ik}(0)\right] < 0. \end{aligned} \tag{3.39}$$

for all values of $t \geqslant 0$

Moreover, (3.36) and (3.37) are automatically satisfied at the initial time $t=0$; therefore, a system of gravitating bodies satisfying condition (3.39) will satisfy (3.36) and (3.37) for all values of $t \geqslant 0$. Thus, the distances between the bodies tend to infinity as $t \to \infty$.

It remains for us to investigate conditions (3.39).

All the integrals which appear in (3.39) are of the same type, namely, for any values of i and j we have

$$\int_0^t \frac{dt}{f_{ij}} = \int_0^t \frac{dt}{a_{ij}t^2 + b_{ij}t + c_{ij}}, \tag{3.40}$$

where

$$a_{ij} = \left[\frac{2}{3} v_{ij}(0) - \frac{1}{3} r'_{ij}(0)\right]^2, \tag{3.41}$$

$$b_{ij} = \left[\frac{2}{3} r'_{ij}(0) - \frac{1}{3} v_{ij}(0)\right] 2r_{ij}(0), \tag{3.42}$$

$$c_{ij} = r^2_{ij}(0). \tag{3.43}$$

It is well known that the properties and the evaluation of any integral of the form (3.40) essentially depend on the sign of the discriminant

$$\Delta_{ij} = 4a_{ij}c_{ij} - b^2_{ij}.$$

Using the values of a_{ij}, b_{ij}, and c_{ij} given by (3.41) (3.43), as well as conditions (3.35), we find after some simple computations that

$$\Delta_{ij} = \frac{4}{3} r_{ij}^2(0)\left[v_{ij}^2(0) - r_{ij}'^2(0)\right] > 0. \qquad (3.44)$$

For $\Delta_{ij} > 0$ and $a_{ij} > 0$, the quadratic form $a_{ij}t^2 + b_{ij}t + c_{ij}$ is positive for all values of t. Therefore,

$$\int_0^t \frac{dt}{a_{ij}t^2 + b_{ij}t + c_{ij}} < \int_0^\infty \frac{dt}{a_{ij}t^2 + b_{ij}t + c_{ij}}. \qquad (3.45)$$

Furthermore, when $\Delta_{ij} > 0$ and, as in our case, $a_{ij} > 0$, we have

$$\int_0^\infty \frac{dt}{a_{ij}t^2 + b_{ij}t + c_{ij}} =$$

$$= \frac{\pi}{\sqrt{\Delta_{ij}}} - \frac{2}{\sqrt{\Delta_{ij}}} \operatorname{arc\,tan} \frac{b_{ij}}{\sqrt{\Delta_{ij}}}.$$

Since $b_{ij} < 0$, as follows from equation (3.42) and conditions (3.35), from the last equation we obtain the estimate

$$\int_0^\infty \frac{dt}{a_{ij}t^2 + b_{ij}t + c_{ij}} < \frac{2\pi}{\sqrt{\Delta_{ij}}}. \qquad (3.46)$$

Finally, from (3.40), (3.44), (3.45), and (3.46), we find that the inequalites

$$\int_0^t \frac{dt}{f_{ij}} < \frac{2\pi}{\frac{1}{\sqrt{3}} r_{ij}(0) \sqrt{v_{ij}^2(0) - r_{ij}'^2(0)}}. \qquad (3.47)$$

hold for any values of i and j and for all $t \geqslant 0$. From this last result, we immediately obtain the estimate

$$m_k \int_0^t \frac{dt}{r_{ik}} + \sum_{\substack{j=0\\ j\neq i,\,k}}^{n-1} m_j \int_0^t \frac{dt}{r_{ij}} + \sum_{\substack{j=1\\ j\neq k}}^{n-1} m_j \int_0^t \frac{dt}{r_{jk}} <$$

$$< \frac{\pi (M - m_i)(M - m_k)}{\frac{1}{\sqrt{3}} r_{ik}(0) \sqrt{v_{ik}^2(0) - r_{ik}'^2(0)}}, \tag{3.48}$$

where $M = \sum_{i=0}^{n-1} m_i$.

On the basis of (3.48), we can state that if the inequalities

$$\frac{\pi (M - m_i)(M - m_k)}{\frac{1}{\sqrt{3}} r_{ik}(0) \sqrt{v_{ik}^2(0) - r_{ik}'^2(0)}} -$$

$$-\frac{1}{3}[v_{ik}(0) + r_{ik}'(0)] < 0,$$

hold at the initial time $t = 0$, then (3.39) and all other consequences of it will hold for all values of $t \geqslant 0$.

To conclude our investigation, we can formulate the following theorem.

THEOREM 3.6. *If the initial conditions*

$$v_{ik}(0) - 2r_{ik}'(0) > 0,$$

$$v_{ik}^2(0) - r_{ik}'^2(0) > 0,$$

$$[v_{ik}(0) + r_{ik}'(0)] \sqrt{v_{ik}^2(0) - r_{ik}'^2(0)} > \frac{3\sqrt{3}\,\pi (M - m_i)(M - m_k)}{r_{ik}(0)}$$

are satisfied at $t=0$ *by a system of gravitating bodies, then the distances* r_{ik} *between the bodies tend to infinity as* $t \to \infty$.

We shall now make some comments apropos of the theorem just proved.

The criterion we have obtained for the complete scattering of a system of bodies as $t \to \infty$ requires each of the velocities $v_{ik}(0)$ to be sufficiently great in comparison with both the reciprocal of the distance $r_{ik}(0)$ and the radial velocity $r'_{ik}(0)$. The first requirement is intrinsic to the problem, whereas the second is a consequence of the mathematical reasoning used in the derivation of the criterion.

The theorem imposes the fewest conditions on the initial state of the system when $r'_{ik}(0)=0$. We note, in passing, that it is just in this very case that the criterion of Theorem 3.4 fails to answer the basic question. When $r'_{ik}(0)=0$, the first and second conditions of Theorem 3.6 are automatically satisfied and the third simply becomes

$$v_{ik}^2(0) > \frac{3\sqrt{3}\,\varkappa\,(M-m_i)(M-m_k)}{r_{ik}(0)}$$

If the relative and radial velocities are equal, namely, if $v_{ik}(0)=r'_{ik}(0)$, then Theorem 3.6 fails to produce an answer. However, in this case, Theorem 3.4 is applicable and the criterion established there is the most suitable one. Thus Theorems 3.4 and 3.6 complement one another.

Finally, we note that the criterion obtained above may be regarded as a weakened variant of the criterion for hyperbolic motion formulated by G.A. Merman (see the reference 1953a) for the particular case of the three-body problem. We would like to remark on this variant in the

following connection. The criterion of Merman is not very transparent, and its form does not permit us to tell whether the set of initial conditions satisfying this criterion is not empty. In our case, this is obviously so. Moreover, it is not clear under what conditions Merman's criterion is applicable; however, they cannot differ essentially from those under which Theorem 3.6 holds.

9. Let us now consider a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$. We describe the motion of the system using Jacobi coordinates. Set $\rho = \rho_{n-1}$, where ρ_{n-1} denotes the distance of P_{n-1} from the center of mass of the remaining bodies. Let $\boldsymbol{\rho}$ denote the vector joining the center of mass of the system $P_0, P_1, \ldots, P_{n-2}$ to the body P_{n-1}. Then

$$\boldsymbol{\rho} = \mathbf{i}\xi_{n-1} + \mathbf{j}\eta_{n-1} + \mathbf{k}\zeta_{n-1}, \tag{3.49}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors directed along the ξ_{n-1}, η_{n-1}, and ζ_{n-1} axes, respectively. Differentiating (3.49) twice with respect to time, we obtain

$$\boldsymbol{\rho}'' = \mathbf{i}\xi''_{n-1} + \mathbf{j}\eta''_{n-1} + \mathbf{k}\zeta''_{n-1},$$

or, by the equations of motion (1.7),

$$\boldsymbol{\rho}'' = \frac{1}{\mu_{n-1}}\left\{\mathbf{i}\frac{\partial U}{\partial \xi_{n-1}} + \mathbf{j}\frac{\partial U}{\partial \eta_{n-1}} + \mathbf{k}\frac{\partial U}{\partial \zeta_{n-1}}\right\}. \tag{3.50}$$

Now, using the relations

$$\frac{\partial U}{\partial \xi_{n-1}} = -\sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \cdot \frac{\partial r_{i,n-1}}{\partial \xi_{n-1}},$$

$$\frac{\partial U}{\partial \eta_{n-1}} = -\sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \cdot \frac{\partial r_{i,n-1}}{\partial \eta_{n-1}},$$

$$\frac{\partial U}{\partial \zeta_{n-1}} = -\sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \cdot \frac{\partial r_{i,n-1}}{\partial \zeta_{n-1}},$$

we can write (3.50) in the form

$$\rho'' = -\frac{1}{\mu_{n-1}} \sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \left\{ \mathbf{i} \frac{\partial r_{i,n-1}}{\partial \xi_{n-1}} + \mathbf{j} \frac{\partial r_{i,n-1}}{\partial \eta_{n-1}} + \mathbf{k} \frac{\partial r_{i,n-1}}{\partial \zeta_{n-1}} \right\}.$$

Integrating this equation with respect to time from $t=0$ to $t>0$, we obtain

$$\rho' - \rho'(0) =$$

$$= -\frac{1}{\mu_{n-1}} \int_0^t \sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \left\{ \mathbf{i} \frac{\partial r_{i,n-1}}{\partial \xi_{n-1}} + \mathbf{j} \frac{\partial r_{i,n-1}}{\partial \eta_{n-1}} + \mathbf{k} \frac{\partial r_{i,n-1}}{\partial \zeta_{n-1}} \right\} dt,$$

Using the fact that

$$\left| \mathbf{i} \frac{\partial r_{i,n-1}}{\partial \xi_{n-1}} + \mathbf{j} \frac{\partial r_{i,n-1}}{\partial \eta_{n-1}} + \mathbf{k} \frac{\partial r_{i,n-1}}{\partial \zeta_{n-1}} \right| < 1,$$

we find from this that

$$|\rho' - \rho'(0)| \leqslant \frac{1}{\mu_{n-1}} \int_0^t \sum_{i=0}^{n-2} \frac{m_i m_{n-1}}{r_{i,n-1}^2} \, dt. \qquad (3.51)$$

If it is assumed that no a-approach of P_{n-1} to the other bodies occurs for $t \geqslant 0$ (the definition of a-approach was given in Section 4 of this chapter), then

$$r_{i,n-1} > a\rho.$$

Using this to strengthen inequality (3.49), we then obtain

$$|\rho' - \rho'(0)| < \frac{M^*}{a^2}\int_0^t \frac{dt}{\rho^2}, \tag{3.52}$$

where

$$M^* = \frac{1}{\mu_{n-1}}\sum_{i=0}^{n-2} m_i m_{n-1}.$$

The integral inequality (3.52) is of the same type as those in the system of integral inequalities (3.33) considered in the preceding section. Applying the second Theorem on the Inductiveness of Inequalities to (3.52) and repeating the argument used to prove Theorem 3.6, we arrive at the following theorem.

THEOREM 3.7. *If the initial conditions*

$$v(0) - 2\rho'(0) > 0,$$

$$v^2(0) - \rho'^2(0) > 0,$$

$$|v(0) + \rho'(0)|\sqrt{v^2(0) - \rho'^2(0)} > \frac{3\sqrt{3}\,\pi M^*}{a^2\rho(0)}$$

are satisfied at $t=0$ *by a system of gravitating bodies, where*

$$v(t) = \sqrt{\xi'^2_{n-1} + \eta'^2_{n-1} + \zeta'^2_{n-1}},$$

and if no α-approach of P_{n-1} to the other bodies of the system occurs for any value of $t \geqslant 0$, then $\rho(t) \to \infty$ as $t \to \infty$. Therefore, the distance of P_{n-1} from each of the other bodies will become infinite as $t \to \infty$.

10. We shall say that a system of gravitating particles is *completely dissipative* as $t \to \infty$ if all the distances between the particles increase indefinitely as $t \to \infty$.

THEOREM 3.8. *If at the initial moment of time $t=0$, the conditions*

$$\rho'(0) > 0,$$

$$\rho'^2(0) > \max\left\{\frac{4M}{\alpha^2(0)}, \frac{8H}{\mu_{n-1}}\right\}, \tag{3.53}$$

are satisfied by a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$, where H is the total energy of the system, and if there occurs no α-approach of P_{n-1} to the other bodies for any value of $t \geqslant 0$, then the distance of P_{n-1} from each of the other bodies increases indefinitely as $t \to \infty$, but the system of bodies P_0, $P_1, \ldots, P_{n-1}$ on the whole is not completely dissipative.

That the distance of P_{n-1} from each of the other bodies in the system increases indefinitely as $t \to \infty$ follows immediately from Theorem 3.5a.

We must now show that the system of bodies in question cannot be completely dissipative as $t \to \infty$.

From the energy integral

$$U = \frac{1}{2}\sum_{i=1}^{n-1} \mu_i \left(\xi_i'^2 + \eta_i'^2 + \zeta_i'^2\right) - H$$

we have

$$U \geqslant \frac{1}{2}\mu_{n-1}\left(\xi'^2_{n-1}+\eta'^2_{n-1}+\zeta'^2_{n-1}\right)-H,$$

and using the obvious inequality

$$\xi'^2_{n-1}+\eta'^2_{n-1}+\zeta'^2_{n-1}\geqslant \rho'^2(t),$$

we find that

$$U \geqslant \frac{1}{2}\mu_{n-1}\rho'^2(t)-H.$$

From Theorem 3.5a it follows that $\rho'(t)>\frac{1}{2}\ \rho'(0)$ for all values of $t\geqslant 0$. Therefore,

$$U > \frac{1}{8}\mu_{n-1}\rho'^2(0)-H.$$

Hence by (3.53) we have

$$U>0$$

and, therefore, U does not approach zero. This means the system of gravitating bodies P_0, P_1, . . ., P_{n-1} cannot be completely dissipative as $t\to\infty$.

11. In proving some of the theorems, namely, Theorems 3.5, 3.5a, 3.7, and 3.8, we assumed there occurred no α-approach of P_{n-1} (whose motion we were studying) to the other bodies P_0, P_1, . . ., P_{n-2}. It seems to us that the presence of such a condition in the problem is due to its intrinsic nature.

In fact, consider the aggregate of the gravitating bodies P_0, P_1, . . ., P_{n-1}, and suppose that P_{n-1} is located inside the aggregate. If the initial velocity of P_{n-1} is sufficiently great, and if while moving inside the aggregate P_{n-1} does not approach the other bodies closely, then P_{n-1} will leave

the aggregate and go off to infinity. In other words, its distance from each of the other bodies will become infinite as $t \to \infty$.

However, if P_{n-1} comes close to the other bodies, this could result in such an exchange of energy between the bodies in the system, that P_{n-1}, by having its velocity changed, could not leave the aggregate in spite of the fact its initial velocity was sufficient to have this happen under the other conditions.

However, it is necessary to observe still one other aspect of the problem. It could happen that two of the bodies P_i and P_{n-1} are ejected from the aggregate, and in moving an infinite distance away from the other bodies of the system, are within an α-proximity of one another or else arrive at this state after a certain time has elapsed. In this case, the theorems we have proved fail to solve our problem.

Thus, the criteria formulated in Theorems 3.5, 3.5a, 3.7, and 3.8 yield a solution to the problem only if the body in question experiences no subsequent α-approaches to the other bodies. Since these theorems do not indicate under what conditions any α-approach is excluded, they merely have qualitative significance. Nevertheless, they can be used quantitatively in some problems in the dynamics of stellar systems. For example, by knowing the parameters of a stellar system, it is possible to estimate the probability of an α-approach of a given body to any of the other bodies and, therefore, to estimate the probability of occurrence of the type of motions we have been considering, under given conditions on the initial state of the system.

Finally, we note that the condition that there be no α-approaches can be omitted in certain instances since it is automatically satisfied. Namely, this is so in the

application of Theorem 3.8 to the particular case of the three-body problem. It was in this way that the author first obtained criteria for hyperbolic-elliptic motions in the three-body problem. Later on, G. A. Merman, using our method of continuous induction, gave several interesting theorems containing more exact criteria for hyperbolic-elliptic motions in the three-body problem.

CHAPTER 4

Method of Invariant Measure

1. The application of the method of invariant measure to mechanical problems and, in particular, to the many-body problem is based on the phase interpretation of mechanical systems.

The phase interpretation of a system of n gravitating bodies is as follows. In Chapter 1, we saw that the second-order differential equations of motion can be replaced by an appropriate number of first-order equations, the general form of which may be written as

$$\begin{aligned} \frac{dx_1}{dt} &= X_1(x_1, \ldots, x_\nu), \\ &\cdots\cdots\cdots \\ \frac{dx_\nu}{dt} &= X_\nu(x_1, \ldots, x_\nu) \end{aligned} \tag{4.1}$$

The state of the system of bodies is then completely defined at each moment of time t by the values of ν real continuous functions of t, $x_1, x_2, \ldots, x_\nu$. We call each possible set of values of $x_1, x_2, \ldots, x_\nu$ a *phase* of the dynamical system. While regarding the collection of numbers $x_1, x_2, \ldots, x_\nu$ as the phase of the dynamical system, we shall, at the same time, think of it as a point in ν-dimensional euclidean space E^ν, which we refer to as *phase space*.

We now choose some moment of time as our initial time which we designate to be $t=0$. Each point $p \in E^\nu$, with the exception of those which are singular points of

the given system, is a possible phase of the dynamical system at time $t=0$. The phase of the system at any time t other than the initial time will in general be different from the initial phase. We let $f(p,t)$ denote the point of E^ν that represents this new phase, thus $f(p,0)=p$. As the time varies positively, $f(p,t)$ will be defined in some instances for all positive values of t, whereas in other instances, there will be a $\tau>0$ such that $f(p,t)$ will be defined only for all positive $t<\tau$. A similar situation will prevail when the time varies negatively.

As the time varies, the phase $x_1, x_2, \ldots, x_\nu$ will change in accordance with the laws inherent in the motion of the mechanical system. Corresponding to these changes of phase, the point $f(p,t)$ will move in a given way in E^ν and will describe a certain trajectory.

By simultaneously considering all the points of phase space and their motions, we obtain the picture of the flow of the "phase-point fluid." This picture describes the aggregate of possible motions of the dynamical system. By applying the method of invariant measure, we shall be able to characterize the motion of the particles in physical space in terms of the motion of the points of multidimensional phase space.

2. For each pair of points of the space E^ν

$$p'=(x_1', x_2', \ldots, x_\nu'),$$
$$p''=(x_1'', x_2'', \ldots, x_\nu'')$$

we introduce the nonnegative number

$$\rho(p', p'')=\sqrt{\sum_{i=1}^{\nu}(x_i'-x_i'')^2}\,.$$

called the distance between p' and p''. The general properties of the distance in E^{ν} are exactly the same as those in three-dimensional space:

(1) (Identity property) $\rho(p',p'')=0$ if and only if $p' = p''$;

(2) (Symmetry property) $\rho(p',p'')=\rho(p'',p')$;

(3) (Triangle property) For any three points p',p'',p''' of E^{ν}, we have

$$\rho(p', p'')+\rho(p'', p''')\geqslant \rho(p', p''').$$

Let p be a point and A an arbitrary set of E^{ν}. The nonnegative number

$$\rho(p, A)=\inf_{q\in A} \rho(p, q)$$

is called the distance between the point p and the set A.

Using the notion of distance, it is easy to introduce into E^{ν} the notions of the neighborhood of a point and the neighborhood of a set. If p is any point of E^{ν} and ϵ is an arbitrary positive number, then the set of all points p' for which $\rho(p,p')<\epsilon$ is called a spherical neighborhood of p of radius ϵ and is denoted by $S(p,\epsilon)$. If A is any set of E^{ν}, then the set

$$S(A, \epsilon)=\bigcup_{p\in A} S(p, \epsilon)$$

is called a spherical neighborhood of A of radius ϵ.

Although it is impossible to give a graphical representation of the motion of a point and its trajectory, or the motion of a set, in phase space, nevertheless, both the above interpretation with its geometrical terminology and the analogy of the properties of the space E^{ν} to those of ordinary three-dimensional space considerably facilitate the discussion.

Since distances and neighborhoods are characterized in ν-dimensional space by properties analogous to those of distances and neighborhoods in ordinary spaces, the theory of point sets and the theory of measure can be suitably generalized to the space E^ν. A presentation of these theories and, in particular, the basic concepts that we shall use may be found in books on the theory of sets and the theory of functions of a real variable.

3. The general properties of the phase motions $f(p,t)$ will, of course, depend on the character of the functions $X_i(x_1, \ldots, x_\nu)$ that appear on the right-hand side of (4.1). For a wide class of equations of the form (4.1), to which the equations of motion of a system of gravitating bodies also belong, it can be shown that the phase motions have the following properties:

(1) *Uniqueness property*—every point p of phase space (with the exception of singular points) defines a unique motion of $f(p,t)$ in an interval of time of one of the following four types:

$$\begin{aligned} -\infty &< t < +\infty, \\ -\infty &< t < t'', \\ t' &< t < +\infty, \\ t' &< t < t''. \end{aligned}$$

Thus, through each point of phase space passes only one phase trajectory.

(2) *Stationarity property*—in the interval in which $f(p,t)$ is defined, the condition

$$f(f(p,\ t_1),\ t_2) = f(p,\ t_1 + t_2);$$

is satisfied.

(3) *Continuous dependence of the motion on time*—for every t in the interval of time in which $f(p,t)$ is defined and

for any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\rho(f(p,\ t),\ f(p,t')) < \epsilon$$

for all t' such that

$$|\ t - t'\ | < \delta;$$

(4) *Continuous dependence of the motion on the initial point*—if the motion of $f(p,t)$ is defined over the interval $[t_1,t_2]$, then for $\epsilon > 0$ we can choose a positive δ sufficiently small so that for any point q for which $f(q,t)$ is defined and satisfies the condition

$$\rho(f(q,\ t_1),\ f(p,\ t_1)) < \delta,$$

the motion of $f(p,t)$ exists in the interval $[t_1,t_2]$ and satisfies there the condition

$$\rho(f(q,\ t),\ f(p,\ t)) < \epsilon.$$

4. Of great importance below will be the concept of the measure of a set of points of phase space. We shall use the measure normally defined in multidimensional euclidean space by the direct generalization of the concept of measure in three-dimensional space. This measure is only meaningful for bounded sets. However, along with bounded sets, we are going to consider unbounded meassurable sets both of finite and of infinite measure. To our end, it suffices to define the measure of an unbounded set as follows. Let $R_1, R_2, \ldots, R_n, \ldots$ be an unbounded monotonic increasing sequence of positive numbers. Let S_n denote the sphere of radius R_n with center at the origin. We call an unbounded set A measurable if each of the sets $A \cap S_n$ is measurable. The sequence meas$(A \cap S_n)$, $n = 1,2, \ldots$, either tends to a limit or becomes unbounded as $n \to \infty$. If the limit exists, we then define

$$\text{meas } A = \lim_{n \to \infty} \text{meas } (A \cap S_n),$$

and in the contrary case, we set

$$\text{mes } A = \infty$$

Clearly, when this definition is applied to a bounded set, we obtain its usual measure.

The great importance of the concept of measure is in the fact that Liouville's Theorem on Invariant Measure is applicable to the motion of a system of gravitating bodies in phase space. The invariance of measure in phase space is the basis that allows us to ascertain some of the very important properties of dynamical systems.

THEOREM 4.1. (Liouville's Theorem) *If the equations*

$$\begin{aligned} \frac{dx_1}{dt} &= X_1(x_1, \ldots, x_\nu), \\ &\cdots\cdots\cdots \\ \frac{dx_\nu}{dt} &= X_\nu(x_1, \ldots, x_\nu) \end{aligned} \tag{4.1}$$

are such that

$$\sum_{i=1}^{\nu} \frac{\partial X_i}{\partial x_i} = 0, \tag{4.2}$$

and if A is a measurable set of E^ν of finite measure such that for any point $p \in A$ the motion of $f(p,t)$ is defined for all $t \geqslant 0 (t \leqslant 0)$, *then*

$$\text{meas } f(A, t) = \text{meas } A$$

for all $t \geqslant 0 (t \leqslant 0)$, i.e., *the measure is invariant when* $t \geqslant 0 (t \leqslant 0)$.

Proof: Let A be a measurable set of E^ν of finite measure, where for any $p \in A$, the motion of $f(p,t)$ is defined for all $t \geqslant 0 (t \leqslant 0)$:

$$\text{meas}\, f(A, t) = \int\limits_{f(A,t)} dx_1 \ldots dx_\nu. \tag{4.3}$$

Consider the solution of (4.1)

$$\begin{aligned} x_1 &= x_1(t;\ a_1,\ \ldots,\ a_\nu), \\ &\cdots\cdots\cdots\cdots \\ x_\nu &= x_\nu(t;\ a_1,\ \ldots,\ a_\nu), \end{aligned} \tag{4.4}$$

satisfying the initial condition

$$\left.\begin{aligned} x_1 &= a_1 \\ &\cdots \\ x_\nu &= a_\nu \end{aligned}\right\} \quad \text{for} \quad t = 0,$$

where $(a_1, \ldots, a_\nu)$ is any point of the set A.

Let us introduce new variables of integration $\bar{x}_1, \ldots, \bar{x}_\nu$ defined by equations analogous to (4.4):

$$\begin{aligned} x_1 &= x_1(t;\ \bar{x}_1,\ \ldots,\ \bar{x}_\nu), \\ &\cdots\cdots\cdots\cdots \\ x_\nu &= x_\nu(t;\ \bar{x}_1,\ \ldots,\ \bar{x}_\nu), \end{aligned} \tag{4.5}$$

in which t is regarded as a parameter. In terms of the new variables, (4.3) becomes

$$\text{meas}\, f(A, t) = \int\limits_A D \cdot d\bar{x}_1 \ldots d\bar{x}_\nu, \tag{4.6}$$

where D is the Jacobian

$$D = \frac{\partial(x_1, \ldots, x_\nu)}{\partial(\bar{x}_1, \ldots, \bar{x}_\nu)}.$$

Differentiating (4.6) with respect to the parameter t, we obtain

$$\frac{d}{dt}\operatorname{meas} f(A, t) = \int\limits_A \frac{\partial D}{\partial t}\, d\bar{x}_1 \, \ldots \, d\bar{x}_\nu,$$

Then going back to the old variables $x_1, \ldots, x_\nu$ we find that

$$\frac{d}{dt}\operatorname{meas} f(A, t) = \int\limits_{f(A, t)} \frac{1}{D}\frac{\partial D}{\partial t}\, dx_1 \, \ldots \, dx_\nu. \tag{4.7}$$

By the rule for differentiating a determinant,

$$\frac{\partial D}{\partial t} = \sum_{i=1}^{\nu} D_i, \tag{4.8}$$

where

$$D_i = \frac{\partial\left(x_1, \ldots, x_{i-1}, \dfrac{\partial x_i}{\partial t}, x_{i+1}, \ldots, x_\nu\right)}{\partial(\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \ldots, \bar{x}_\nu)}.$$

Or by (4.1)

$$\ldots x_{i-1}, X_i, x_{i+1}, \ldots, x_\nu) \;/\; \ldots \bar{x}_\nu).$$

$$D_i = \sum_{k=1}^{\nu} \frac{\partial X_i}{\partial x_k} \frac{\partial (x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, x_\nu)}{\partial (\dot{x}_1, \ldots, \dot{x}_{i-1}, \dot{x}_i, \dot{x}_{i+1}, \ldots, \dot{x}_\nu)}$$

By then noting that

$$\frac{\partial (x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, x_\nu)}{\partial (\dot{x}_1, \ldots, \dot{x}_{i-1}, \dot{x}_i, \dot{x}_{i+1}, \ldots, \dot{x}_\nu)} = \begin{cases} D & \text{for } k = i \\ 0 & \text{for } k \neq i \end{cases},$$

we find

$$D_i = D \frac{\partial X_i}{\partial x_i}.$$

Substituting this expression for D_i in (4.8), we obtain

$$\frac{\partial D}{\partial t} = D \sum_{i=1}^{\nu} \frac{\partial X_i}{\partial x_i},$$

and hence

$$\frac{1}{D} \frac{\partial D}{\partial t} = \sum_{i=1}^{\nu} \frac{\partial X_i}{\partial x_i}.$$

Using this last result, we can express formula (4.7) in the following form:

$$\frac{d}{dt} \operatorname{meas} f(A, t) = \int\limits_{f(A, t)} \sum_{i=1}^{\nu} \frac{\partial X_i}{\partial x_i} \, dx_1 \ldots dx_\nu. \quad (4.9)$$

This is the remarkable formula of Liouville.

We now proceed to prove Liouville's Theorem. From formula (4.9) and condition (4.2), we have

$$\frac{d}{dt} \text{meas } f(A, t) = 0.$$

Integrating this with respect t from $t=0$ up to $t>0$ and taking into account that $f(A,0)=A$, we find that

$$\text{meas } f(A, t) = \text{meas } A.$$

The theorem is proved.

It is necessary to emphasize the importance of the requirement that for any point $p \in A$, the motion of $f(p,t)$ is defined for all $t \geqslant 0 (t \leqslant 0)$. If this requirement is not satisfied, the set $f(A,t)$ and its measure may not be meaningful for all values of $t \geqslant 0 (t \leqslant 0)$. This can even be true for differential equations of a very simple form as the following examples show.

EXAMPLE 1. Consider the motion defined by the equations

$$\frac{dx}{dt} = e^x, \qquad \frac{dy}{dt} = -ye^x. \tag{4.10}$$

Phase space for this system is the euclidean plane E^2. From (4.10), we find that

$$\frac{dy}{dx} = -y, \qquad y = Ce^{-x},$$

where C is the constant of integration. Thus the trajectories of the phase motions are exponential curves. Let us choose an arbitrary point p in the plane with coordinates x_0, y_0, and let us consider the motion of $f(p,t)$. Integrating the first of equations (4.10) subject to the initial condition: $x = x_0$ when $t=0$, we find that

$$x = -\ln(e^{-x_0} - t).$$

From this result it is easy to see that $x \to \infty$ as $t \to e^{-x_0}$. Therefore, the phase point $f(p,t)$ goes off to infinity in a finite interval of time, and for all $t \geqslant e^{-x_0}$, the motion of $f(p,t)$ is not defined. This is true for any choice of the initial point $p \in E^2$.

EXAMPLE 2. Consider the motion defined by the equations

$$
\begin{aligned}
\frac{dx}{dt} &= \sqrt{2\pi}\, e^{\frac{x^2}{2}}, \\
\frac{dy}{dt} &= -\sqrt{2\pi}\, xy e^{\frac{x^2}{2}}.
\end{aligned} \tag{4.11}
$$

Phase space for this system is again the euclidean plane E^2. From (4.11), we find that

$$\frac{dy}{dx} = -xy,$$

$$y = Ce^{-\frac{x^2}{2}},$$

where C is the constant of integration. Thus the trajectories of the phase motions are normal curves. Let us choose an arbitrary point p in the plane with coordinates x_0, y_0, and let us consider the motion of $f(p,t)$. Integrating the first of equations (4.11) subject to the initial condition: $x = x_0$ when $t = 0$, we find that

$$t = \frac{1}{\sqrt{2\pi}} \int_{x_0}^{x} e^{-\frac{x^2}{2}}\, dx.$$

From this we see that

$$x \to +\infty \quad \text{as} \quad t \to \frac{1}{\sqrt{2\pi}} \int_{x_0}^{+\infty} e^{-\frac{x^2}{2}}\, dx < 1,$$

$$x \to -\infty \quad \text{as} \quad t \to \frac{1}{\sqrt{2\pi}} \int_{x_0}^{-\infty} e^{-\frac{x^2}{2}}\, dx > -1.$$

Thus, the phase point $f(p,t)$ goes off to infinity in a finite interval of time when the time varies both positively and negatively. In this example, every motion is defined in an interval of the form

$$t' < t < t'',$$

where $t' \geqslant -1$ and $t'' \leqslant 1$.

5. The main object of study in general dynamics are the invariant sets of phase space. A set A is called *positive* (*negative*) *invariant* or *invariant as* $t \to \infty$ $(t \to -\infty)$ if for every point $p \in A$, $f(p,t)$ is defined for all $t \geqslant 0$ $(t \leqslant 0)$, and for these times

$$f(p,t) \in A.$$

A set A is called simply invariant if it is invariant both as $t \to \infty$ and as $t \to -\infty$. It is clear that the union or intersection of any collection of sets that are invariant either as $t \to \infty$ or as $t \to -\infty$, as well as the difference of two invariant sets, is also invariant as $t \to \infty$ or as $t \to -\infty$.

Let us consider the set of all points $p \in E^r$ for which the motion of $f(p,t)$ is defined for all $t \geqslant 0$ $(t \leqslant 0)$, and let Ξ^+ (Ξ^-) denote this set. Obviously, Ξ^+ (Ξ^-) is a maximal set, invariant as $t \to \infty$ $(t \to -\infty)$; we call it the *regular kernel* of the dynamical system for $t \to \infty$ $(t \to -\infty)$.

We call the set

$$\Xi = \Xi^+ \cap \Xi^- \tag{4.12}$$

simply the regular kernel of the dynamical system.

THEOREM 4.2. *Ξ^+, Ξ^-, and Ξ are sets of type G_δ, i.e., sets which are the intersection of a countable number of open sets.*

We first prove the theorem for the set Ξ^+. Let

$$\tau_1, \tau_2, \ldots, \tau_n, \ldots$$

be an unbounded monotonic increasing sequence of positive numbers as $n \to \infty$. Let Ξ_k^+ denote the set of all points p of phase space for which the motion of $f(p,t)$ is defined in the interval $[0,\tau_k]$. If $p \in \Xi_k^+$, then due to the continuous dependence of the phase motion on the initial point, we may choose a positive number δ_k such that for every point q, with $q \in S(p, \delta_k)$, the motion of $f(q,t)$ is defined for all $t \in [0,\tau_k]$. However, this means that $S(q, \delta_k) \cancel{\subset} \Xi_k^+$, and therefore Ξ_k^+ is an open set. Obviously,

$\subseteq$

$$\Xi^+ = \bigcap_{k=1}^{\infty} \Xi_k^+,$$

from which it follows that Ξ^+ is a set of type G_δ and is therefore measurable.

The theorem is proved in an analogous way for Ξ^-.

The validity of the theorem for the set Ξ follows immediately from (4.12).

6. Let us take some set A of E^ν at the initial moment of time. Each point $p \in A$ will move in its own way in E^ν, and at some other moment of time, all the points p of A will occupy new positions $f(p,t)$ and form a new set; we denote this set by $f(A,t)$.

Let p and A be such that $f(p,t)$ and $f(A,t)$ are defined for all $t \geqslant 0 (t \leqslant 0)$. From the definition of $f(p,t)$ and $f(A,t)$. it follows that if $p \in A$, then $f(p,t) \in f(A,t)$ for every

$t \geqslant 0 (t \leqslant 0)$. Conversely, if $f(p, t) \in f(A, t)$ for some $t \geqslant 0 (t \leqslant 0)$, then $p \in A$.

From this remark, we immediately have the following simple facts. If A_1 and A_2 are sets such that $A_1 \cap A_2 = 0$, and if $f(A_1, t)$ and $f(A_2, t)$ are defined for all $t \geqslant 0 (t \leqslant 0)$, then for these values of t we have

$$f(A_1, t) \cap f(A_2, t) = 0.$$

On the other hand, if A_1 and A_2 are sets such that $A_1 \cap A_2 = 0$, and if $f(A_1 t)$ and $f(A_2, t)$ are defined for all $t \geqslant 0 (t \leqslant 0)$ then for these values of t we have

$$f(A_1, t) \cap f(A_2, t) \neq 0.$$

A set A is called *recurrent as* $t \to \infty$ $(t \to -\infty)$ if for any $T > 0$, we can find a $t' > T$ such that

$$A \cap f(A, t') \neq 0 \quad (A \cap f(A, -t') \neq 0).$$

Conversely, the set A is *not recurrent as* $t \to \infty$ $(t \to -\infty)$ if we can find a $\tau > 0$ such that

$$A \cap f(A, t) = 0 \quad (A \cap f(A, -t) = 0)$$

for all $t > \tau$.

Theorem 4.3. *If a set A is recurrent as $t \to \infty$ $(t \to -\infty)$, and if $f(p, t)$ is defined for all $t < 0 (t > 0)$ and for any point $p \in A$, then A is recurrent as $t \to -\infty$ $(t \to \infty)$.*

If A is not recurrent as $t \to \infty (\to -\infty)$, and if $f(p, t)$ is defined for all $t < 0 (t > 0)$ and for any point $p \in A$, then A is not recurrent as $t \to -\infty$ $(t \to \infty)$.

Proof: Let the set A be recurrent as $t \to \infty$. Then, for any positive T no matter how large, we can find a $t' > T$ such that

$$A \cap f(A, t') \neq 0.$$

We then have

$$f(A,-t')\cap f(f(A,t'),-t')\neq 0,$$
$$f(A,-t')\cap A\neq 0,$$

and this implies that A is recurrent as $t\mapsto-\infty$.

For a set A which is recurrent as $t\mapsto-\infty$, the theorem is proved analogously.

The second part of the theorem is easily shown by arguing by contradiction.

THEOREM 4.4. *If a set A is such that $f(A,t)$ is defined for all real values of t, and if we can find a number $\tau>0$ such that $A\cap f(A,t)=0$ or $A\cap f(A,-t)=0$ for all $t\geqslant\tau$, then the sets*

$$\ldots, f(A,-n\tau),\ldots,f(A,-\tau),A,f(A,\tau),\ldots,f(A,n\tau),\ldots$$

are pairwise disjoint.

We prove the theorem by contradiction, and we assume that we can find two integers i and j, with $i<j$, such that

$$f(A,i\tau)\cap f(A,j\tau)\neq 0. \tag{4.13}$$

Consider the sets A and $f(A,(j\text{-}i)\tau)$; since $j\text{-}i\geqslant 1$, we have by assumption

$$A\cap f(A,(j-i)\tau)=0.$$

and, therefore,

$$f(A,i\tau)\cap f(f(A,(j-i)\tau),i\tau)=0,$$

However,

$$f(f(A,(j-i)\tau),i\tau)=f(A,j\tau).$$

Thus

$$f(A,i\tau)\cap f(A,j\tau)=0,$$

and this contradicts (4.13), and so the theorem is proved.

The theorem is proved analogously for the case where $A \cap f(A, -t) = 0$ for all $t \geqslant \tau$.

7. A point p is called *recurrent as* $t \to \infty$ $(t \to -\infty)$ if for any positive T no matter how large and any $\epsilon > 0$, there exists a time $t' > T$ $(t' < -T)$ such that

$$f(p, t') \in S(p, \epsilon).$$

A point p is called simply *recurrent* if it is recurrent both when $t \to \infty$ and when $t \to -\infty$.

THEOREM 4.5. *If the point p is recurrent as $t \to \infty$ $(t \to -\infty)$, then for any $t_1 > 0$ $(t_1 < 0)$, the point $f(p, t_1)$ is also recurrent as $t \to \infty$ $(t \to -\infty)$.*

We shall only prove the theorem for a point p which is recurrent as $t \to \infty$. The case where the point is recurrent as $t \to -\infty$ is treated analogously.

Let us take arbitrary positive numbers ϵ and T, and let us choose $\delta = \delta(\epsilon, T)$ sufficiently small so that

$$f(S(p, \delta)\, t_1) \subset S(f(p, t_1), \epsilon).$$

This is possible because of the continuous dependence of the motion on the initial point. Since the point p is recurrent as $t \to \infty$, we can find a $t' > T$ such that

$$f(p, t') \in S(p, \delta).$$

If q is a point of the set A, then by the definition of $f(q, t)$ and $f(A, t)$ it follows that $f(q, t) \in f(A, t)$. Using this fact, we can write

$$f(f(p, t'), t_1) \in f(S(p, \delta), t_1) \subset S(f(p, t_1), \epsilon),$$

and since

$$f(f(p, t'), t_1) = f(f(p, t_1), t'),$$

it follows that

$$f\,(f\,(p,\,t_1),\,t')\in S\,(f\,(p,\,t_1),\,\epsilon).$$

This means that the point $f(p,t_1)$ is recurrent as $t\to\infty$. We thus see that if p is recurrent as $t\to\infty$, then all points $f(p,t)$ will be recurrent as $t\to\infty$. Therefore it is natural to call the motion of $f(p,t)$ recurrent as $t\to\infty$. The same remarks are also valid for points that are recurrent as $t\to-\infty$.

8. Consider the set P of all recurrent points in the phase space of a mechanical system. For the following, it will be important to establish its measurability. We shall also need to consider the set $P^+(P^-)$ of all points of phase space that are recurrent as $t\to\infty$ (or $t\to-\infty$).

THEOREM 4.6. *P^+, P^-, and P are sets of type G_δ and hence are measurable.*

Proof: Let

$$\tau_1,\ \tau_2,\ \ldots,\ \tau_n,\ \ldots$$

be an unbounded monotonic increasing sequence of positive numbers and

$$\eta_1,\ \eta_2,\ \ldots,\ \eta_n,\ \ldots$$

a monotonic decreasing sequence of positive numbers which tends to zero as a limit. Let P_n^+ denote the set of points p for which one can find a $\tau>\tau_n$ such that $\rho(p,\ f(p,\tau))<\eta_n$. Consider a point $p'\in P_n^+$, and let $\rho(p',f(p',\tau))=\beta<\eta_n$. We then choose a positive number $\epsilon<(\eta_n-\beta)/2$ and a $\delta>0$ sufficiently small so that for any point q for which $\rho(p',q)<\delta$, the inequality $\rho(f(p',t),f(q,t))<\epsilon$ is satisfied for all positive $t\leqslant\tau$. This is possible because of the continuous dependence of the phase motion on the initial point. We then have

$$\rho(q, p') < \delta \leqslant \varepsilon < {}^1/_2(\eta_n - \beta),$$
$$\rho(p', f(p, \tau)) = \beta,$$
$$\rho(f(p', \tau),\ f(q, \tau)) < \varepsilon < {}^1/_2(\eta_n - \beta),$$

and hence by the Triangle Property, it follows that

$$\rho(q, f(q, \tau)) < \eta_n.$$

This means that from the condition $p' \in P_n^+$, it follows that

$$S(p', \varepsilon) \subset P_n^+,$$

i.e., P_n^+ is an open set. Obviously

$$P^+ = \bigcap_{n=1}^{\infty} P_n^+,$$

and this implies that P^+ is a set of type G_δ.

The fact that P^- is also a set of type G_δ is proved analogously.

The validity of the theorem for the set P then follows from the obvious relation

$$P = P^+ \cap P^-.$$

COROLLARY. *The sets E'—P and Ξ—P are measurable.*

9. The theorems presented above were of a preliminary nature. We now proceed to prove the main theorems.

THEOREM 4.7. *Let $\mathfrak{M}$ be a measurable set not containing points that are recurrent as $t \to -\infty$), where for each $p \in \mathfrak{M}$, $f(p,t)$ is defined for all $t \geqslant 0$ (or $t \leqslant 0$). If meas $\mathfrak{M} > 0$, then one can find a bounded set $\mathfrak{M}^* \subset \mathfrak{M}$ of positive measure which is not recurrent as $t \to \infty$ (or $t \to -\infty$).*

We shall only prove the theorem for $t \to \infty$. The case $t \to -\infty$ is treated analogously.

Let

$$\tau_1, \tau_2, \ldots, \tau_n, \ldots$$

be an unbounded monotonic increasing sequence of positive numbers, and

$$\eta_1, \eta_2, \ldots, \eta_n, \ldots$$

a monotonic decreasing sequence of positive numbers tending to the limit zero. We let $\mathfrak{M}_n$ denote the points $p \in \mathfrak{M}$ such that

$$f(p, t) \bar{\in} S(p, \eta_n)$$

for all $t \geqslant \tau_n$. Since the set $\mathfrak{M}$ does not contain any recurrent points, each point $p \in \mathfrak{M}$ belongs to at least one of the $\mathfrak{M}_n$ and, therefore,

$$\mathfrak{M} = \bigcup_{n=1}^{\infty} \mathfrak{M}_n. \tag{4.14}$$

Let us show that $\mathfrak{M}_n$ is measurable. To do this, we shall prove that each of the sets $\mathfrak{M}_n$ is closed with respect to $\mathfrak{M}$. Let p^* be a point of $\mathfrak{M}$ which is a limit point for $\mathfrak{M}_n$. Let us assume that $p^* \bar{\in} \mathfrak{M}_n$; then we can find a time $t_1 > \tau_n$ such that

$$\rho(p^*, f(p^*, t_1)) = \beta < \eta_n.$$

Now let $\epsilon < (\eta_n - \beta)/2$, and let us choose $\delta > 0$ sufficiently small so that for any point q for which $\rho(p^*, q) < \delta$, the inequality

$$\rho(f(p^*, t_1), f(q^*, t_1)) < \epsilon,$$

is satisfied. This is possible because of the continuous dependence of the phase motion on the initial point. Moreover, the point q may be chosen in $\mathfrak{M}_n$ since p^* is a limit point of $\mathfrak{M}_n$. Therefore,

$$\rho(q,\ p^*) < \delta \leqslant \varepsilon < \frac{1}{2}(\eta_n - \beta),$$

$$\rho(p^*,\ f(p^*,\ t_1)) = \beta,$$

$$\rho(f(p^*,\ t_1),\ f(q,\ t_1)) < \varepsilon < \frac{1}{2}(\eta_n - \beta),$$

or by the Triangle Property

$$\rho(q,\ f(q,\ t_1)) < \eta_n,$$

This contradicts the fact that q is in $\mathfrak{M}_n$. Therefore, $p^* \in \mathfrak{M}_n$ and $\mathfrak{M}_n$ is closed.

From (4.14), it follows that

$$\sum_{n=1}^{\infty} \text{meas } \mathfrak{M}_n \geqslant \text{meas } \mathfrak{M} > 0,$$

and, therefore, we can find a set $\mathfrak{M}_k$ such that

$$\text{meas } \mathfrak{M}_k > 0.$$

Let π be a point of metric density of the set $\mathfrak{M}_k$, and consider the set

$$\mathfrak{M}_k' = \mathfrak{M}_k \cap S\left(\pi,\ \frac{1}{2}\eta_k\right).$$

It is clear that $\mathfrak{M}_k'$ is measurable and that meas $\mathfrak{M}_k' > 0$. The diameter of $\mathfrak{M}_k'$ does not exceed $\eta_k/2$. But for our point p of $\mathfrak{M}_k'$, we have

$$\rho(p, f(p, t) >)\eta_k$$

for all $t > \tau_k$. Therefore,

$$\mathfrak{M}'_k \cap f(\mathfrak{M}'_k, t) = 0$$

for $t > \tau_k$. Setting $\mathfrak{M}^* = \mathfrak{M}'_k$ completes the proof of the theorem.

THEOREM 4.8. *If A is a measurable set in E^ν such that any subset of it of positive measure is recurrent as $t \to \infty$ (or $t \to -\infty$), then all points of A, with the possible exception of a set of measure zero, are recurrent as $t \to \infty$ (or $t \to -\infty$).*

We prove the theorem by contradiction, and we assume that there exists a set $\mathfrak{M}$ of positive measure all of whose points are not recurrent as $t \to \infty$ (or $t \to -\infty$). Then by Theorem 4.7, we can find a bounded measurable set $\mathfrak{M}^*$ of positive measure which is not recurrent as $t \to \infty$ (or $t \to -\infty$). However, this contradicts the conditions of the theorem.

Let A be a measurable but otherwise arbitrary set of phase space of the dynamical system. We call the set A *regular as $t \to \infty$* if for any $\tau > 0$, we can find a set A^* such that meas $A^* < \infty$ and

$$\bigcup_{0 \leqslant t \leqslant \tau} f(A, t) \subset A^*.$$

A set which is regular as $t \to -\infty$ is defined in a similar way.

Let B be a set and p a point of phase space; we call the point p *recurrent with respect to the set B as $t \to \infty$ ($t \to -\infty$)* if for any $T > 0$, we can find a positive (or negative) number t', with $|t'| > T$, such that $f(p, t') \in B$.

THEOREM 4.9. *Let A be a regular set of points of phase space with invariant measure and W a set whose points are*

recurrent with respect to A *as* $t \to \infty$ *(or* $t \to -\infty$*). Then any measurable non-recurrent set* $\mathfrak{M}^* \in W$ *is a set of measure zero.*

We prove the theorem by contradiction, and we assume that there exists a set $\mathfrak{M}^* \subset W$ which is not recurrent and which has positive measure. Then we can give a positive number τ such that the sets

$$f(\mathfrak{M}^*, n\tau), \quad n = 0, 1, \ldots$$

are pairwise disjoint. Since A is regular, there exists a set A^*, with meas $A^* < \infty$, such that

$$\bigcup_{0 \leqslant t \leqslant \tau} f(A, t) \subset A^*.$$

It is not difficult to see that for any $p \in \mathfrak{M}^*$, we can choose a subsequence from the sequence $f(p, n\tau)$, $n = 0, 1, 2, \ldots$, all of whose terms are points in A^*.

Let us consider the sets

$$\mathfrak{M}_n^* = f(\mathfrak{M}^*, n\tau), \quad n = 0, 1, 2 \ldots$$

and let

$$D_n = A^* \cap \mathfrak{M}_n^*.$$

Each D_n is measurable, being the intersection of two measurable sets. Moreover, the sets D_n are disjoint. Therefore the series

$$\sum_{\nu=0}^{\infty} \text{meas} D_\nu \tag{4.15}$$

converges since

$$D_\nu \subset A^*,$$

$$\bigcup_{\nu=0}^{\infty} D_\nu \subset A^*,$$

$$\text{meas} \bigcup_{\nu=0}^{\infty} D_\nu \leqslant \text{meas} A^* < \infty$$

and

$$\sum_{\nu=0}^{\infty} \text{meas} D_\nu = \text{meas} \bigcup_{\nu=0}^{\infty} D_\nu < \infty. \tag{4.16}$$

Further,

$$f(D_n, -n\tau) = \mathfrak{M}^* \cap f(A^*, -n\tau) \tag{4.17}$$

is the set of points p in $\mathfrak{M}^*$ for which we have

$$f(p, n\tau) \in A^*.$$

Each point of $\mathfrak{M}^*$ belongs to an infinite number of the sets (4.17) and, therefore, to each of the sets

$$\mathfrak{N}_n = \bigcup_{\nu=n}^{\infty} f(D_\nu, -\nu\tau). \tag{4.18}$$

$$n = 1, 2, \ldots$$

Thus, for any n we have

$$\mathfrak{M}^* \subset \mathfrak{N}_n. \tag{4.19}$$

By the invariance of measure, we obtain

$$\text{meas } f(D_\nu, -\nu\tau) = \text{meas } D_\nu, \tag{4.20}$$

and so by (4.18) and (4.20)

$$\text{meas } \mathfrak{N}_n = \sum_{\nu=n}^{\infty} \text{meas} f(D_\nu, -\nu\tau) = \sum_{\nu=n}^{\infty} \text{meas } D_\nu.$$

Now, the series (4.15) converges and

$$\sum_{\nu=n}^{\infty} \text{meas } D_\nu$$

is its remainder. Thus for any $\epsilon > 0$, we can choose an integer $N = N(\epsilon)$ sufficiently large that

$$\text{meas } \overline{\mathfrak{M}}_n < \varepsilon \tag{4.21}$$

for all $n \geqslant N$.

Taking into account relation (4.19), and inequality (4.21), we conclude that

$$\text{meas } \mathfrak{M}^* < \varepsilon,$$

and since ϵ is an arbitrary positive number, this implies that

$$\text{meas } \mathfrak{M}^* = 0.$$

We have arrived at a contradiction, and this proves the theorem.

10. We now study the motion of a system of n gravitating bodies in phase space making use of the analytical relations of celestial mechanics and the above general measure theoretical results on dynamical systems.

Let us first consider the set Q of all points $p \in \Xi$ for which

$$\max_{ij} \{r_{ij}\} \rightarrow \infty \quad \text{as} \quad t \rightarrow -\infty,$$

$$\max_{ij} \{r_{ij}\} \not\rightarrow \infty \quad \text{as} \quad t \rightarrow +\infty.$$

We are not going to investigate the measure of Q. For our purposes, it will suffice to prove that Q satisfies the following purely logical theorem.

THEOREM 4.10. *From the assumption that*

$$\text{meas } Q > 0,$$

it follows that all the points of Q, with the possible exception of a set of measure zero, satisfy the condition

$$\inf_{t \geqslant 0}(\min_{ij}\{r_{ij}\}) = 0.$$

Proof: Let Q^- denote the set of all points $p \in E^{6n}$ for which

$$r^*(t) = \max\{r_{ij}\} \to \infty \quad \text{as} \quad t \to -\infty, \qquad (4.22)$$

and let us show that this set is measurable. From condition (4.22) it follows that for any point $p \in Q^-$ and any number $R > 0$, we can find a positive number T so large that

$$r^*(t) \geqslant R \text{ for all } t < -T.$$

Let

$$R_1, R_2, \ldots, R_n, \ldots$$

$$T_1, T_2, \ldots, T_n, \ldots$$

be two unbounded monotonic increasing sequences of positive numbers. We let $Q^-\{R_k, T_m\}$ denote the set of all points for which

$$r^*(t) \geqslant R_k \text{ for all } t < -T_m$$

We now show that this set is closed.

Consider the set G_k of all points $p \in E^{6n}$ for which

$$r^*(p) < R_k;$$

since $r^*(p)$ is a continuous function of the phase point,

G_k is an open set. Now arguing by contradiction, we assume that $Q^-\{R_k,T_m\}$ is not closed. Then we can find a point $p^*\bar{\in} Q^-\{R_k,T_m\}$ which is a limit point of this set. For this point, we can find a moment of time $t'<-T_m$ such that

$$f(p^*,\ t')\in G_k$$

Since the set G_k is open, it Is possible to find a number $\epsilon>0$ sufficiently small that

$$S(f(p^*,\ t'),\ \varepsilon)\subset G_k.$$

Due to the continuous dependence of the phase motion on the initial point, we can choose a $\delta>0$ sufficiently small that

$$f(S(p^*,\ \delta),\ t')\subset S(f(p^*,\ t'),\ \varepsilon)\subset G_k. \quad (4.23)$$

However, p^* is a limit point of the set $Q^-\{R_k,T_m\}$ and therefore we can find a point p' such that

$$\begin{gathered} p'\in Q^-\{R_k,\ T_m\}, \\ p'\in S(p^*,\ \delta). \end{gathered} \quad (4.24)$$

From (4.23) and (4.24), it follows that

$$f(p',\ t')\in G_k,$$

which leads to a contradiction. Thus the set $Q^-\{R_k,T_m\}$ is closed.

Let us next consider the set

$$Q_k^-=\bigcup_{m=1}^{\infty}(Q^-\{R_k,\ T_m\}\cap\Xi). \quad (4.25)$$

Its measurability follows from the results obtained above.

Now, Q_k^- is the set of points of phase space each of which eventually leaves the set

$$r^* < R_k \tag{4.26}$$

permanently as $t \to -\infty$.

From the obvious formula

$$Q^- = \bigcap_{k=1}^{\infty} Q_k^-$$

it follows that the set Q^- is measurable. The set Q^- contains the points of phase space that each eventually leaves any set of the form

$$r^* (p) < R_n, \atop n=1, 2, \ldots \tag{4.27}$$

permanently as $t \to -\infty$.

Carrying out a similar argument for $t \to \infty$, we can construct measurable sets

$$Q_k^+ = \bigcup_{m=1}^{\infty} (Q^+ \{R_k, T_m\} \cap \Xi),$$

analogous to the sets (4.25). Then

$$Q^- \cap (\Xi - Q_k^+)$$

is a measurable set consisting of the points in phase space each of which eventually leaves any set of the form (4.27) permanently as $t \to -\infty$ and which is also recurrent with respect to the set (4.26) as $t \to \infty$.

It is further clear that

$$Q = \bigcup_{k=1}^{\infty} (Q^- \cap (\Xi - Q_k^+)),$$

from which it follows first of all that Q is measurable.

We are assuming meas $Q>0$, and therefore, we can find a ν such that for the set

$$Q_\nu = Q^- \cap (\Xi - Q_\nu^+) \tag{4.28}$$

we have

$$\text{meas } Q_\nu > 0.$$

It is very important to note for the following that Q_ν is the set of points each of which, as $t \to -\infty$, eventually leaves any set of the form

$$r^*(p) \leqslant R_n, \quad n=1, 2, \ldots$$

permanently, and as $t \to \infty$ is recurrent with respect to the set

$$r^*(p) \leqslant R_\nu.$$

Let

$$a_1, a_2, \ldots, a_n, \ldots$$

be a monotonic sequence of positive numbers which converges to zero, and let Λ_n denote the set of points p in $E^6{}_n$ for which

$$r_*(p) = \min\{r_{ij}\} \geqslant a_n \text{ for all } t \geqslant 0.$$

Let us show that each Λ_n is closed. Consider the set G_k' of all points for which

$$r_*(p) < a_k;$$

since $r_*(p)$ is a continuous function of the phase point, G_k' is open. Let us assume that Λ_k is not closed. Then we can find a point $p_* \overline{\in} \Lambda_k$ which is also a limit point of

this set. For this point, we can then find a time $t'' > 0$ such that

$$f(p_*, t'') \in G'_k.$$

Since G'_k is an open set, we can find an $\epsilon > 0$ sufficiently small such that

$$S(f(p_*, t'')) \subset G_k.$$

By the continuous dependence of the phase motion on the initial point we can choose a $\delta > 0$ so small that

$$f(S(p_*, \delta), t'') \subset S(f(p_*, t''), \epsilon) \subset G'_k. \quad (4.29)$$

Now, p_* is a limit point of Λ_n. Therefore, we can find a point p'' such that

$$\begin{aligned} p'' &\in \Lambda_k, \\ p'' &\in S(p_*, \delta), \end{aligned} \quad (4.30)$$

From (4.29) and (4.30) it follows that

$$f(p'', t'') \in G'_k,$$

and this leads to a contradiction. The set Λ_k is closed and hence measurable.

We now consider the proof of the theorem itself.

We argue by contradiction. If the theorem is not true, then we can find a μ such that for the measurable set

$$Q^{\mu}_{\nu} = Q_{\nu} \Lambda_{\mu},$$

we have

$$\text{meas}\, Q^{\mu}_{\nu} > 0. \quad (4.31)$$

Let us take an unbounded monotonic sequence of positive numbers

$$H_1, H_2, \dots, H_n, \dots,$$

and let us denote by W_k the set of all points whose phase coordinates satisfy the inequality

$$|H| = \left| \frac{1}{2} \sum_{i=1}^{n} m_i \left(x_i'^2 + y_i'^2 + z_i'^2 \right) - U \right| \leqslant H_k.$$

Since the left-hand side of this inequality is a function of the phase and continuous on Ξ, each of the W_k is an open set. Writing Q_μ'' as the sum of a denumerable number of measurable sets

$$Q_\nu^\mu = \bigcup_{k=1}^{\infty} (Q_\nu^\mu \cap W_k)$$

and taking into account inequality (4.31), we conclude that we can find a positive integer h such that for the set

$$W = Q_\nu^\mu \cap W_h,$$

we have

$$\text{meas } W > 0.$$

For the following, it will be important to enumerate the properties of the points p in the set W:

(1) if $p \in W$, then as $t \to -\infty$, $f(p,t)$ leaves any set of the form

$$r^* (p) \leqslant R_n$$
$$n = 1, 2, \dots$$

permanently, i.e., $r^* \to (t) \infty$ as $t \to -\infty$;

(2) every point $p \in W$ is recurrent with respect to the set

$$r^* (p) < R_\nu$$

as $t \to \infty$;

(3) if $p \in W$, then the motion of $f(p,t)$ is such that $r_*(t) \geqslant \alpha_\mu$ for all $t \geqslant 0$;

(4) if $p \in W$, then the motion of $f(p,t)$ satisfies the inequality

$$|H| = \left| \frac{1}{2} \sum_{i=1}^{n} m_i \left(x_i'^2 + y_i'^2 + z_i'^2 \right) - U \right| \leqslant H_h$$

for all real values of t.

We let B denote the set of points $p \in \Xi$ for which

$$\min \{r_{ij}\} \geqslant \alpha_\mu \text{ for all } t \geqslant 0. \tag{4.32}$$

$$|H| < H_h; \tag{4.33}$$

From the preceding considerations, it follows that this set is measurable. Moreover, it is clear that if $p \in W$, then

$$f(p,\, t) \in B \text{ for all } t \geqslant 0 \tag{4.34}$$

so that B is invariant as $t \to \infty$. Let us show that the phase velocity v, considered as a function of the points of phase space, is bounded on the set B. The square of this velocity is given by

$$v^2 = \sum_{i=1}^{n} \left\{ \left(\frac{dx_i'}{dt} \right)^2 + \left(\frac{dy_i'}{dt} \right)^2 + \left(\frac{dz_i'}{dt} \right)^2 + \left(\frac{dx_i}{dt} \right)^2 + \left(\frac{dy_i}{dt} \right)^2 + \left(\frac{dz_i}{dt} \right)^2 \right\} =$$

$$= \sum_{i=1}^{n} \left\{ \left(\frac{1}{m_i} \frac{\partial U}{\partial x_i} \right)^2 + \left(\frac{1}{m_i} \frac{\partial U}{\partial y_i} \right)^2 + \left(\frac{1}{m_i} \frac{\partial U}{\partial z_i} \right)^2 + x_i'^2 + y_i'^2 + z_i'^2 \right\}. \tag{4.35}$$

From inequality (4.32), it follows that the potential func-

tion U is bounded from above; therefore we can find an $H^*>0$ such that $U \leqslant H^*$, and hence we have

$$\frac{1}{2}\sum_{i=1}^{n} m_i\left(x_i'^2+y_i'^2+z_i'^2\right)<H+H^*,$$

i.e., the kinetic energy is bounded from above. Thus the functions $x_i'^2$, $y_i'^2$, and $z_i'^2$ are bounded from above. Differentiating U with respect to x_i, we obtain

$$\frac{\partial U}{\partial x_i}=-\sum_{ij}\frac{m_i m_j}{r_{ij}^2}\cdot\frac{\partial r_{ij}}{\partial x_i},$$

Since

$$\left|\frac{\partial r_{ij}}{\partial x_i}\right| \leqslant 1,$$

we have that

$$\left|\frac{\partial U}{\partial x_i}\right| \leqslant \sum_{ij}\frac{m_i m_j}{r_{ij}^2}.$$

From this inequality and condition (4.32), it follows that

$$\left(\frac{\partial U}{\partial x_i}\right)^2$$

is bounded from above. In a similar way, we can show that the functions

$$\left(\frac{\partial U}{\partial y_i}\right)^2,\ \left(\frac{\partial U}{\partial z_i}\right)^2.$$

are bounded. From these results and equation (4.35), it follows that v^2, and therefore also v, is bounded on the set B.

Let

$$A=B\cap S\left(nR_v\right),$$

where $S(nR_\nu)$ is an open sphere with center at the origin, n being a positive integer (we are considering the motion of the n bodies in a coordinate system with origin at the center of mass). We have shown that the phase velocity is bounded on the set B. Therefore, we can find a positive number b such that $|v|<b$. Since $A\in B$, this estimate is also valid for the set A. Let us choose a $\tau>0$ but otherwise arbitrary; for any $p\in B$, the length of the path traversed by the point $|f(p,t)|$ in phase space in the interval of time from 0 to τ does not exceed $b\tau$. Thus every point $p\in A$ is such that for all t in the interval $0\leqslant t\leqslant\tau$, we have

$$f(p,t)\in S(nR_\nu+b\tau).$$

It then follows that

$$\bigcup_{0\leqslant t\leqslant\tau} f(A,\ t)\subset S(nR_\nu+b\tau),$$

and since $S(nR_\nu+b\tau)$ is a bounded set, meas $S(nR_\nu+b\tau)<\infty$ and therefore, A is regular as $t\to\infty$.

To complete the proof of the theorem, we must obtain a contradiction.

Let us first show that there exists a set $\mathfrak{M}^*\subset W$ which is not recurrent as $t\to-\infty$ such that meas $\mathfrak{M}^*>0$. Assume that this is not true. Then every measurable subset of W of positive measure is recurrent. However, in this case, we have by Theorem 4.8 that all points of W, with the possible exception of a subset of measure zero, are recurrent as $t\to-\infty$. For the motion of $f(p,t)$, determined by any point $p\in W$, we have that

$$\max\{r_{ij}\}\to\infty\quad\text{as}\quad t\to-\infty.$$

This is impossible for recurrent points, and we arrive at a contradiction. Thus, there exists a set $\mathfrak{M}^*\subset W$, with meas $\mathfrak{M}^*>0$, which is not recurrent as $t\to-\infty$. By

Theorem 4.3, $\mathfrak{M}^*$ is also not recurrent as $t\to\infty$. Now, we know that each point $p\in W$ is recurrent as $t\to\infty$ with respect to the set

$$r^*(p) < R_\nu.$$

Therefore, if at some moment of time the point $f(p,t')$ belongs to the set $r^*(p)<R_\nu$, the distances between the gravitating bodies will not exceed R_ν. Therefore, the distance of any of the bodies from the origin, located at the center of mass, does not exceed nR_ν, i.e., $f(p,t')\in S(nR_\nu)$. Hence, every point p that is recurrent with respect to the set $r^*(p)<R_\nu$, will be recurrent with respect to $S(nR_\nu)$. These remarks, in conjunction with condition (4.34), allow us to say that every point $p\in W$ will be recurrent as $t\to\infty$ with respect to the set

$$A = B\cap S(nR_\nu),$$

a set that is regular as $t\to\infty$. We have thus arrived at the conditions of Theorem 4.9 from which it follows that meas $\mathfrak{M}^*=0$. Our contradiction is obtained, and the theorem is proved.

We shall call a system of n gravitating bodies *almost stable as* $t\to\infty$ if

$$\max_{ij}\{r_{ij}\} \not\to \infty \quad \text{as} \quad t\to+\infty,$$

and if we can find a number $\alpha>0$ such that the condition

$$\min_{ij}\{r_{ij}\}\geqslant \alpha$$

is satisfied for all $t\geqslant 0$.

We shall call a system of n gravitating bodies *stable as* $t\to\infty$ if we can find positive numbers R and α such that for all $t\geqslant 0$

$$\max_{ij}\{r_{ij}\}<R, \quad \min_{ij}\{r_{ij}\}\geqslant\alpha$$

At present, it is not known whether there exist almost stable systems which at the same time are not stable.

We shall say that a system of n gravitating bodies contains a body which *has come from infinity* if

$$\max_{ij} \{r_{ij}\} \to \infty \quad \text{as} \quad t \to -\infty.$$

THEOREM 4.11. *For all initial conditions, with the possible exception of a set of initial conditions of measure zero, a system of gravitating bodies which contains a body that has come from infinity cannot be almost stable.*

We prove the theorem by contradiction.

If the theorem is not true, we can find a number $\alpha > 0$ such that the set Q^* of all points $p \in \Xi$ satisfying the conditions

$$\begin{aligned} &\max_{ij} \{r_{ij}\} \to \infty \quad \text{as} \quad t \to -\infty, \\ &\max_{ij} \{r_{ij}\} \to \infty \quad \text{as} \quad t \to +\infty \\ &\inf_{t \geq 0} (\min_{ij} \{r_{ij}\}) \geq \alpha \end{aligned} \tag{4.36}$$

is a set of positive measure:

$$\text{meas} \quad Q^* > 0. \tag{4.37}$$

However, $Q^* \subset Q$, where Q is the set of points $p \in \Xi$ satisfying conditions (4.35). From the fact that $Q^* \in Q$ and the condition (4.37), it follows that

$$\text{meas}\, Q > 0. \tag{4.38}$$

Now, by Theorem 4.10, we know that when condition (4.38) is satisfied by all points of Q, with the possible exception of a subset of measure zero, the condition

$$\inf_{t \geqslant 0} (\max_{ij} \{r_{ij}\}) = 0.$$

must be satisfied.

This result, clearly, is also valid for the subset Q^*, and this contradicts condition (4.36).

The theorem is thus proved.

CHAPTER 5

Analysis of Some Cases of the Evolution of a System of Gravitating Bodies

1. The two-century-old development of the planetary cosmogony has often been described as a haphazard alternation of diverse hypotheses, as a process of trial and error that has produced no definite results. This, however, is not so. The development of cosmogony as a science is a process that has followed the general pattern of growth of knowledge. Cosmogonical knowledge has been acquired in the form of relative truths, and as these truths have evolved, we observe the gradual formation of the elements of an objective science, which thus becomes continually more perfect. Of course, the growth of cosmogony, like that of other sciences, has been a very complicated and contradictory thing, in which true and false principles have emerged side by side. However, in the course of its development, certain ideas have appeared that have persevered, and with modification and expansion, have remained as part of the science in one form or another. The history of science shows that such evolving ideas constitute that part of the science which, to the extent of its growth, form the elements of objective knowledge.

In the cosmogony of the planets, one such idea is, first of all, that the solar system and naturally formed systems similar to it evolved out of rotating matter scattered in space. Much controversy arose about the physical state of the matter, about its genetical relationship to the various other cosmical bodies, and about the

nature of the process by which the matter was accumulated and transformed into large bodies. However, a general basis has stood the test of time and is being confirmed by the experience of the two-century-old development of the planetary cosmogony. This basis thus comprises the elements of objective knowledge. The idea that the planets were formed from rotating matter scattered in space follows almost inevitably from the properties of the solar system and the natural laws governing it.

Thus the basic questions we have to consider in cosmogonical investigations are the process by which the matter scattered throughout wide regions of space was accumulated in a small volume, the process by which this matter was condensed, and the process by which tiny particles were unified into large bodies when the cosmical systems were formed. There is no doubt that gravitational forces play an essential role in these phenomena. However, today, it is regarded as well established that gravitational forces alone are insufficient to explain their occurrence, and of fundamental importance also are physical phenomena involving energy conversion. Therefore, the full analysis of these cosmogonical processes is beyond the scope of gravitational mechanics.

Nevertheless, the theorems of celestial mechanics can be of considerable help in these investigations. We begin by considering the following. The rotating matter scattered in space, from which the solar system was formed and which we take as its initial state, i.e., as the object whose evolution is impending, can be regarded as a system of many gravitating bodies. The final product of evolution, the solar system, is also a system of many gravitating bodies which are fewer in number and larger in size. Without going deeply into the processes and mechanism by which the initial system was transformed into the final

one, but by comparing them only as two gravitating systems of bodies, we shall discover that this transformation cannot be realized solely on the basis of the laws of gravitational mechanics. By looking only at the mechanical side of the question, we cannot, to be sure, explain the physical factors which are important to the transformation under consideration. Nevertheless, a consideration of the mechanical side of the question does yield significant results. We may not be able to discover the nature of these physical factors, but we can get some idea about them by determining the *mechanical consequences* of these factors.

2. Let us consider n gravitating bodies $P_0, P_1, \ldots, P_{n-1}$, and let r_{ij} denote the distance between P_i and P_j. We shall say that a *completely hyperbolic approach* of the bodies occurs if all the distances satisfy the condition

$$r_{ij}(t) \to +\infty \quad \text{as } t \to -\infty$$

It is not difficult to see that a completely hyperbolic approach is a type of motion that is actually possible for a system of n gravitating bodies. Namely, we know that in purely dynamical problems, it is legitimate to reverse the sign of the time. Therefore, if the inequalities (3.24) hold for the case of time varying negatively, then from the appropriate reformulation of Theorem 3.4, it will follow that a completely hyperbolic approach takes place. Moreover, since r_{ij} and r'_{ij} are continuous functions of the coordinates of phase space of the system of gravitating bodies, the set of points of phase space whose coordinates satisfy condition (3.24) is an open set. Hence, it follows that the set of points of phase space which are initial points for the "regime" of a completely hyperbolic approach is a set of positive measure.

Let us now show that the motion in which the conditions

$$r_{ij}(t) \to \infty \quad \text{as} \quad t \to -\infty \tag{5.1}$$

and

$$r_{ij}(t) < R \quad \text{as} \quad t \geqslant 0, \tag{5.2}$$

are satisfied simultaneously, where R is any suitable positive number, is impossible for a system of n gravitating bodies.

Disregarding a set of initial states of measure zero (the case $H=0$), we can assert that from (5.1) and Theorem 1.1, it follows that $H>0$. Therefore, by Theorem 1.2 at least one of the distances between the bodies tends to infinity as $t \to \infty$, which contradicts condition (5.2).

We thus see that for a completely hyperbolic approach of gravitating bodies, a system cannot arise in which the mutual distances remain bounded for all $t \geqslant 0$. However, if the bodies approach each other, then the unification of not all, but some portion of them into a subsystem with bounded r_{ij} is still possible. This was rigorously proved for the three-body problem in the theory of capture developed by O. Yu. Schmidt, G. F. Khilmi, N. N. Parisky, and G. A. Merman. For a larger number of bodies, it should be possible to give examples in which such motions occur, however, they would be somewhat artificial and would entail very formidable computations. At any rate, it is possible to get a sufficiently clear picture of our phenomena by means of the following considerations.

If from a number of bodies that have come from infinity, a stable subsystem is isolated following the approach of the bodies, then on the one hand, $U \to 0$ as $t \to -\infty$, and on the other hand, there are distances between the bodies that are bounded on the set $t \geqslant 0$. There-

fore, we can find a constant U_0 and a time T such that $U > U_0$ for $t > T$.

Then from the formula

$$I^2(t) = I_0^2 + I_0^{2\prime} t + 2M \int_0^t \int_0^{t_1} (U + 2H)\, dt_2 dt_1,$$

in which we must take $H > 0$, we see that following the approach of the bodies, the moment of inertia $I^2(t)$ will increase more rapidly as $t \to \infty$ than as $t \to -\infty$ despite the fact that all the distances increase indefinitely as $t \to -\infty$, and only a portion of them increase indefinitely as $t \to \infty$. This means that the gravitational association of a portion of the bodies is accompanied by the rise of motions involving increased velocities.

The physical interpretation of these facts is as follows. The too high kinetic energy which the bodies have as the result of approaching from infinity prevents their complete gravitational association. However, when certain conditions are satisfied, a portion of the bodies can form a stable subsystem. This can happen when the gravitational interaction of the bodies as they approach each other results in a redistribution of the energy causing some parts of the system to begin to move rapidly with respect to the others. If the kinetic energy of the relative motion of parts of the system constitutes a large percentage of the total energy, these parts can then become stable systems. The unification of a portion of the bodies into one or several stable subsystems is necessarily accompanied by the "draining off" of the energy in the system, i.e., by the appearance of relative motions of the subsystems in which the surplus of energy preventing any association of the bodies is spent. Theorem 3.8 shows that if at least one of the bodies in a system of n gravitating bodies has

sufficiently high kinetic energy, the other bodies cannot comprise a dissipative system.

We thus see that the association and scattering of gravitating systems are connected in the closest way. If one of these phenomena occurs, it will be compensated for by the occurrence of the other. *Association and scattering in gravitating systems are thus divergent aspects of the one phenomenon of the gravitational interaction of matter.*

The above results do not mean that whenever we have a hyperbolic approach of many bodies, it is impossible for all of them to be unified into a stable system under their mutual attraction. Our investigations merely point out that this is impossible as a purely dynamical process. However, if we make allowances for processes of a more general type, then this may conceivably happen.

In fact, if whenever the bodies drawing near under their mutual attraction gives rise to or is accompanied by some physical process in which the surplus of kinetic energy preventing their association is absorbed, then association can be realized without the phenomenon of scattering or else with a relaxation of it. In this case, scattering is "replaced" by a physical process that absorbs the surplus of mechanical energy and converts it into non-mechanical form.

For meteoric dust-clouds, the most important form of "drainage" of mechanical energy occurs when a portion of it is converted into thermal energy during non-elastic collisions of the particles. When there is such a "drainage," association is conceivable without the scattering of dust particles or with the relaxation of scattering. In particular, it is possible for meteoric dust particles to be captured when a star passes a meteoric dust-cloud closely.

The dynamics of stellar passages through a cloud of meteoric dust particles was investigated by T. A. Agekyan who worked out a theory of capture tied up to the particles' loss of kinetic energy in mutual nonelastic collisions.

3. We can formulate some further general results by using Theorem 4.11 which was proved in Chapter 4. To do this, we must know under what circumstances the evolution of any particular system of real bodies is describable by means of the dynamical laws for a system of n gravitating bodies.

Real bodies such as the stars, planets, small solid particles, etc., are idealized as mass points. However, the idealization of real bodies as mass points is permissible as long as the distances between them are sufficiently great in comparison to their dimensions. Once the bodies are distributed so closely that the distances between them are comparable to their dimensions this idealization ceases to be correct. In this case, various physical phenomena arise which are accompanied by the conversion of mechanical energy into a non-mechanical form. Finally, when approaching closely, these bodies may undergo considerable changes: they may collide or be disintegrated or, conversely, they may be unified, etc. Therefore, whenever the motion of real bodies is to be described in a given interval of time by the dynamical equations for a system of gravitating bodies, we must assume that the distances between the bodies do not become smaller than some positive number a whose value will depend on the concrete properties of the given system of bodies.

With these remarks, let us consider a system of gravitating bodies $P_0, P_1, \ldots, P_{n-1}$, which we assume to have the following properties:

$$r_{0j}(t) \underset{j=1,2,\ldots,n-1}{\to} \infty \quad \text{as} \quad t \to -\infty,$$

$$r_{ij}(t) \underset{\substack{i,j=1,2,\ldots,n-1 \\ i \neq 1}}{\not\to} \infty \quad \text{as} \quad t \to +\infty,$$

i.e., we consider a system of bodies $P_1, P_2, \ldots, P_{n-1}$ which does not get scattered as $t \to \infty$, to which is adjoined the body P_0 coming in from infinity. We next show that if a physical system is formed after P_0 is adjoined to the system consisting of $P_1, P_2, \ldots, P_{n-1}$ which is stable as $t \to \infty$, then the process by which this system is formed could not be purely mechanical: it is necessarily accompanied by the conversion of mechanical energy into nonmechanical form.

In fact, Theorem 4.11 implies that except for a set of initial conditions of measure zero, for every system of many bodies containing a body incoming from infinity but none outgoing to infinity, the greatest lower bound of the mutual distances r_{ij} is zero. Therefore, arbitrarily close approaches of the bodies in the system will occur, which means also the nonmechanical phenomena that accompany them.

For the sake of simplicity, we have considered the case where the system contains only one body coming in from infinity. However, our conclusions still hold for any number of such bodies.

4. We now derive an inequality which we shall use below.

We consider n gravitating bodies $P_1, P_2, \ldots, P_n$ with masses $m_1, m_2, \ldots, m_n$, and we describe their motion with respect to a set of coordinates with origin at the center of mass of the system.

Consider the following identities whose validity can easily be verified directly:

$$x_i'^2+y_i'^2+z_i'^2=$$
$$=\frac{1}{r_i^2}\left[(x_iy_i'-y_ix_i')^2+(y_iz_i'-z_iy_i')^2+(z_ix_i'-x_iz_i')^2\right]+r_i'^2.$$
$$i=1,2,\ \ldots,\ n$$

Multiplying these identities respectively by m_1, m_2, . . . , and m_n and summing, we obtain

$$2T=P+\Omega^2, \tag{5.3}$$

where T is the kinetic energy and P and Ω^2 are given by

$$P=\sum_{i=1}^{n}\frac{m_i}{r_i^2}(x_iy_i'-y_ix_i')^2+$$

$$+\sum_{i=1}^{n}\frac{m_i}{r_i^2}(y_iz_i'-z_iy_i')^2+\sum_{i=1}^{n}\frac{m_i}{r_i^2}(z_ix_i'-x_iz_i')^2 \tag{5.4}$$

$$\Omega^2=\sum_{i=1}^{n}m_ir_i'^2. \tag{5.5}$$

Now, consider the first sum in the expression on the right-hand side of (5.4). The variables x_i, y_i, x'_i, and y'_i appearing in this sum are not independent. They are connected by relations (1.6) expressing the area integrals. If we use Lagrange multipliers to determine the relative minimum of the first sum in equation (5.4), we find that it is equal to c_1^2/J^2.

In an analogous way, we find that the relative minima of the two other sums on the right-hand side of (5.4) are respectively

$$\frac{c_2^2}{J^2} \text{ and } \frac{c_3^2}{J^2}$$

Using these results and recalling that

$$\Theta^2 = c_1^2 + c_2^2 + c_3^2,$$

we find that

$$P \geqslant \frac{\Theta^2}{J^2}. \tag{5.6}$$

From relations (5.3) and (5.6) it follows that

$$2T \geqslant \frac{\Theta^2}{J^2} + \Omega^2,$$

or

$$J^2 \geqslant \frac{\Theta^2}{2T - \Omega^2}. \tag{5.7}$$

From the energy integral, we have that

$$2T = 2(U+H);$$

substituting this expression for $2T$ in (5.7), we obtain the required inequality

$$J^2 \geqslant \frac{\Theta^2}{2(U+H) - \Omega^2}. \tag{5.8}$$

5. A system of a large number of gravitating bodies is called *stationary*, if despite the movement of its component bodies, the distribution function for the distances between the bodies remains *practically* constant and in this sense is an invariant of the motion. Clearly, a system can be stationary only if its total energy is negative.

If a system is stationary, then

$$J^2 = \text{const}; \quad \frac{d^2 J^2}{dt^2} = 0.$$

Therefore, it follows from the Lagrange-Jacobi equation that

$$U+2H=0.$$

From this equation and the energy integral, we have that

$$T=-H, \quad U=-2H,$$

or taking into account that $H<0$, we can write

$$T=|H|, \quad U=2|H|. \tag{5.9}$$

6. Let a physical system S be in the state (A) at time t_0 in which it is a collection of n_0 mutually gravitating bodies. Suppose that at time $t_1>t_0$, the system S passes into a new state (B) in which it is a system of n_1 gravitating bodies. As to the transition of S from the state (A) to the state (B), we assume nothing other than it has occurred as the result of the gravitational interaction of the moving bodies. Of course, of essential importance in this transition are also the nonmechanical phenomena which are not manifested in the dynamical laws for gravitating bodies. Under this formulation of the problem, the number of bodies n_0 and n_1 in the respective states (A) and (B) are not necessarily equal.

We let

$$T_0, \ U_0, \ |H_0|, \ J_0^2, \ \Theta_0^2, \ \Omega_0^2$$

and

$$T_1, \ U_1, \ |H_1, \ J_1^2, \ \Theta_1^2, \ \Omega_1^2$$

denote the values of the quantities T, U, $|H|$, J^2, Ω^2, and Θ^2 for the respective states (A) and (B) (in other words, at the times t_0 and t_1).

Suppose that state (A) satisfies the condition

$$\left.\frac{d^2J^2}{dt^2}\right|_{(A)} \leqslant 0 \tag{5.10}$$

and suppose that after transition into state (B), S is a stationary system of bodies occupying a region so small that

$$J_1^2 < \frac{\theta_0^2}{2|H_0| - \Omega_0^2}. \tag{5.11}$$

Let us study the transition of S from state (A) to state (B). From the Lagrange-Jacobi equation (1.14) and condition (5.10), it follows that

$$U_0 + 2H_0 \leqslant 0.$$

From this inequality and the energy integral, we obtain

$$-H_0 \geqslant T_0.$$

Taking into account that $H < 0$, we have that

$$|H_0| \geqslant T_0$$

or that

$$2|H_0| \geqslant T_0.$$

Subtracting Ω_0^2 from the left and right-hand sides of this inequality, we obtain

$$2|H_0| - \Omega_0^2 \geqslant T_0 - \Omega_0^2$$

Now, from the definition of T_0 and Ω_0^2, it follows that

$$T_0 - \Omega_0^2 \geqslant 0.$$

and therefore that

$$2|H_0| - \Omega_0^2 > 0.$$

Thus if $\Theta_0^2 > 0$, the following inequality is satisfied:

$$\frac{\Theta_0^2}{2|H_0| - \Omega_0^2} > 0. \tag{5.12}$$

This inequality shows that condition (5.10) is mathematically consistent (admissible).

With the system S in state (B), we now write inequality (5.8) using the condition

$$U_1 = 2|H_1|.$$

which holds because of the stationarity of state (B). We obtain

$$J_1^2 \geqslant \frac{\Theta_1^2}{2|H_1| - \Omega_1^2}. \tag{5.13}$$

From inequalities (5.11) and (5.13), it follows that

$$\frac{\Theta_1^2}{2|H_1| - \Omega_1^2} < \frac{\Theta_0^2}{2|H_0| - \Omega_0^2}.$$

Now, in order for this inequality to hold, at least one of the following inequalities must be satisfied:

$$|H_1| > |H_0|, \tag{5.14}$$

$$\Theta_1^2 < \Theta_0^2, \tag{5.15}$$

$$\Omega_1^2 < \Omega_0^2. \tag{5.16}$$

The results obtained, of course, do not permit us to clarify the character of the non-mechanical processes that

occur in the transition of system S from state (A) to state (B). However, some information concerning these processes can, nevertheless, be obtained. Namely, it is possible to indicate some of the mechanical consequences of the physical phenomena that accompany the transition of S from state (A) to state (B).

Thus, let us consider inequalities (5.14), (5.15), and (5.16). If inequality (5.14) is satisfied when S passes from state (A) into state (B), this means that the transition is accompanied by phenomena that cause a decrease in the mechanical energy of the system. Thus, for example, if the system in the state (A) is a cloud of solid particles, then the nonelastic collisions between these particles will cause a decrease in the mechanical energy of the system since as the result of these collisions a portion of the kinetic energy is converted into heat.

If, however, inequality (5.15) is satisfied, this means that the transition from state (A) into state (B) is accompanied by phenomena for which the moment of orbital momentum of the bodies comprising the system S in the state (A) passes into the moment of axial rotation of the bodies which make up S in the state (B). The transition of the orbital moment into the axial moment is possible if there are collisions between the bodies, or if many bodies are unified into a single larger body.

We must still consider the case where inequality (5.16) is satisfied. If all the orbits of the bodies in the system were circular, $\Omega^2=0$. As the orbits deviate more from circles, i.e., as the radial velocities r' become larger, the value of Ω^2 will become larger.

Therefore, we can regard Ω^2 as a measure of the deviation of the actual orbits from circular orbits. But then the fact that (5.16) holds, means that the transition of S from state (A) to state (B) is accompanied by a

rounding off of the orbits, i.e., by the orbits changing into a more circular shape. This can occur as the result of the interactions and disintegrations of the bodies when they collide, or when many bodies are unified into a single larger body.

It is understood, of course, that one cannot exclude the possibility that the phenomena may be so related that after the transition of the system S from state (A) to state (B), all three inequalities (5.14), (5.15), and (5.16) will be satisfied.

The analysis of the problem in question by means of the methods of celestial mechanics does not permit us to obtain more concrete results. To this end, it is necessary to carry out further cosmogonical investigations. However, we must point out that good agreement does exist between our results and those given by present-day cosmogony. The investigations of academician 0. Yu. Schmidt, confirmed later on by the work of L. E. Gurevich, A. I. Lebedinskii and other research workers, have shown that the formation of the planetary system from scattered matter rotating about the sun was accompanied by conversions of the kinetic energy of this matter into thermal energy, as well as by the unification of the scattered matter into the planets, and by the rounding out of their orbits. The conversion of mechanical energy into nonmechanical form turned out to be the basic process, and the two other processes its consequences.

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